

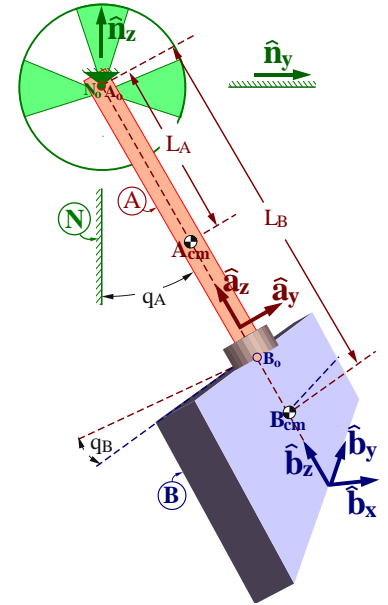
Motivating example (MIPSI): Babyboot

Model

Shown right is a mechanical model of a swinging babyboot (uniform plate B) attached by a shoelace (thin uniform rod A) to a fixed rigid support N . Rod A is attached to N by a revolute joint at point N_o of N . B is attached to A with a second revolute joint at point B_o so B can rotate freely about A 's axis.

Note: The revolute joints' axes are *perpendicular*, not parallel.

- **Bodies:** The rod and plate are **rigid** (inflexible/undeformable).
- **Connections:** The revolute joints are **ideal** (massless, frictionless, with no slop or flexibility).
- **Force:** Earth's gravity is uniform and constant. Other contact forces (e.g., air resistance, solar/light pressure) and distance forces (e.g., electromagnetic, other gravitational) are negligible.
- **Newtonian reference frame:** Earth



Identifiers

Right-handed sets of unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$; $\hat{a}_x, \hat{a}_y, \hat{a}_z$; $\hat{b}_x, \hat{b}_y, \hat{b}_z$ are fixed in N, A, B , respectively, with $\hat{n}_x = \hat{a}_x$ parallel to the revolute axis joining A to N , \hat{n}_z vertically-upward, $\hat{a}_z = \hat{b}_z$ parallel to the rod's long axis (and the revolute axis joining B to A), and \hat{b}_z perpendicular to plate B .

Quantity	Symbol	Type	Value
Earth's gravitational constant	g	Constant	9.81 m/s ²
Distance between N_o and A_{cm}	L_A	Constant	7.5 cm
Distance between N_o and B_{cm}	L_B	Constant	20 cm
Mass of A	m^A	Constant	0.01 kg
Mass of B	m^B	Constant	0.1 kg
A 's moment of inertia about A_{cm} for \hat{a}_x	I^A	Constant	0.05 kg*cm ²
B 's moment of inertia about B_{cm} for \hat{b}_x	I_x^B	Constant	2.5 kg*cm ²
B 's moment of inertia about B_{cm} for \hat{b}_y	I_y^B	Constant	0.5 kg*cm ²
B 's moment of inertia about B_{cm} for \hat{b}_z	I_z^B	Constant	2.0 kg*cm ²
Angle from \hat{n}_z to \hat{a}_z with $+\hat{n}_x$ sense	q_A	Dependent variable	Varies
Angle from \hat{a}_y to \hat{b}_y with $+\hat{a}_z$ sense	q_B	Dependent variable	Varies
Time	t	Independent variable	Varies

Physics

Physics from www.MotionGenesis.com ⇒ [Get Started](#) ⇒ Chaotic Pendulum (Babyboot).

The ODEs (ordinary differential equations) governing the motion of this mechanical system are⁹

$$\ddot{q}_A = \frac{2 \dot{q}_A \dot{q}_B \sin(q_B) \cos(q_B) (I_x^B - I_y^B) - (m^A L_A + m^B L_B) g \sin(q_A)}{I^A + m^A L_A^2 + m^B L_B^2 + I_x^B \cos^2(q_B) + I_y^B \sin^2(q_B)}$$

$$\ddot{q}_B = \frac{-\dot{q}_A^2 \sin(q_B) \cos(q_B) (I_x^B - I_y^B)}{I_z^B}$$

Simplify and solve

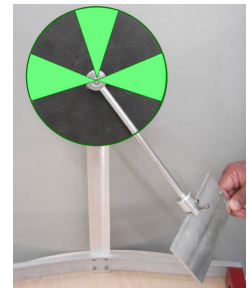
The set of differential equations governing the babyboot's motion are (circle the appropriate qualifiers)

Uncoupled	Linear	Homogeneous	Constant-coefficient	1 st -order
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order

Computers has revolutionized the solution of differential equations. There are a many numerical algorithms for solving nonlinear, coupled, variable coefficient, ODEs (ordinary differential equations) including Euler's method, predictor-corrector, Runge-Kutta, etc. In addition, there are many programs (MATLAB[®], MotionGenesis, WolframAlpha, etc.) that make it easy to solve ODEs.

Computer (numerical) solution of ODEs with MotionGenesis (with plotting)

```
Variable qA'', qB''      % Angles and first/second time-derivatives.
%-----
qA'' = 2*( 508.89*sin(qA) - sin(qB)*cos(qB)*qA'*qB' ) / (-21.556 + sin(qB)^2)
qB'' = -sin(qB)*cos(qB)*qA'^2
%-----
Input tFinal = 10 sec, tStep = 0.02 sec, absError = 1.0E-07
Input qA = 90 deg, qB = 1.0 deg, qA' = 0.0 rad/sec, qB' = 0.0 rad/sec
OutputPlot t sec, qA degrees, qB degrees
%-----
ODE() solveBabybootODE
Quit
```



Alternately: Simplify via linearization and solve analytically (valid only for very small angles)

Linearizing these ODEs about $q_A = 0$ and $q_B = 0$ produces a simpler set of ODEs, namely

$$\ddot{q}_A = -\omega^2 q_A \quad \ddot{q}_B = 0 \quad \text{where } \omega = \sqrt{\frac{(m^A L_A + m^B L_B) g}{I^A + m^A L_A^2 + m^B L_B^2 + I_x^B}}$$

When released from **rest** [i.e., no initial spin, i.e., $\dot{q}_A(0) = \dot{q}_B(0) = 0$], the solutions to these ODEs are

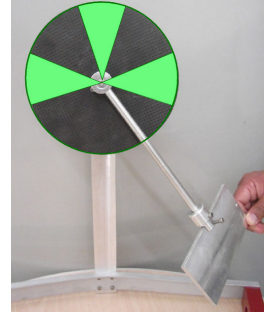
$$q_A(t) = q_A(0) \cos(\omega t) \quad q_B(t) = q_B(0) \quad \text{constant!}$$

⁹Four methods for forming equations of motion are: *Free-body diagrams* of A and B (which is inefficient as it introduces up to 10 unknown force/torque measures); D'Alembert's method (*MG road-maps* of Section ??) which efficiently forms the two equations shown for \ddot{q}_A and \ddot{q}_B (but require a clever selection of systems, points, and unit vectors); *Lagrange's equations* (an energy-based method that automates D'Alembert's cleverness); *Kane's equations* (a modern efficient blend of D'Alembert and Lagrange).

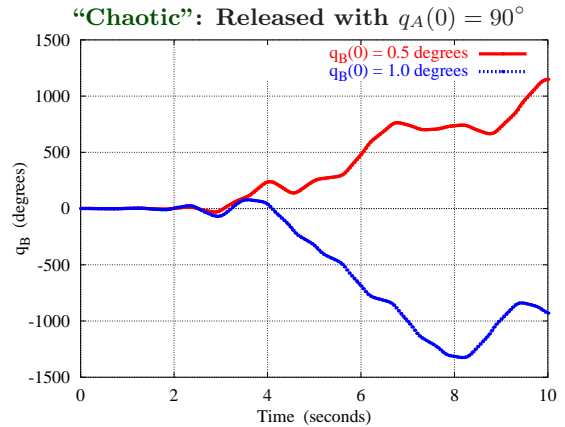
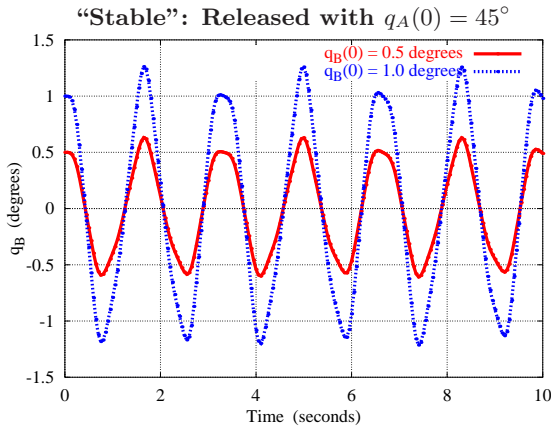
Interpret

This simple system has strange, non-intuitive motion. For certain initial values of q_A , plate B 's motion is well-behaved and “stable”. Alternately, for other initial values of q_A , B 's motion is “**chaotic**” – meaning that a small variation in the initial value of q_B or numerical integration inaccuracies lead to dramatically different results (these ODEs are used to test the accuracy of numerical integrators – the plots below required a numerical integrator error of $\text{absError} = 1 \times 10^{-7}$).

The following chart and figure to the right shows this system's regions of stability (black) and instability (green). Notice the “**chaotic**” plot below shows q_B is *very* sensitive to initial values. A 0.5° change in the initial value of $q_B(0)$ results in more than a 2000° difference in the value of $q_B(t=10)$!



Initial value of q_A		Stability	
$0^\circ \leq q_A(0) \leq 71.3^\circ$		Stable	black
$71.4^\circ \leq q_A(0) \leq 111.77^\circ$		Unstable	green
$111.78^\circ \leq q_A(0) \leq 159.9^\circ$		Stable	black
$160.0^\circ \leq q_A(0) \leq 180.0^\circ$		Unstable	green



More information about this problem is in “Mechanical Demonstration of Mathematical Stability and Instability”, *International Journal of Engineering Education (Journal of Mechanical Engineering Education)*, Vol. 2, No. 4, 1974, pp. 45-47, by Thomas Kane. Or visit www.MotionGenesis.com ⇒ [Get Started](#) ⇒ [Chaotic Pendulum \(Babyboot\)](#).

Investigation of stability: More simulation results

