17.13 Roll that spool - overcoming certainty (Wordplay, the crossword Blog of the New York Times)

How do you overcome certainty? At times, the most successful students (or professionals) regard 'the basics" as intuitive, obvious, or remedial - even when they're wrong. Can confidence limit your ability to think or diminish your curiosity for discovery?[4] A simple dynamics puzzle tries to break learning barriers for those "wicked smart" whiz kids.

Spools. You've seen them for things like thread, ribbon, and telephone cables. But, it may have been a while since you've rolled one around. Few get this puzzle right, except seamstresses and the telephone repair guy.

Here is the puzzle. Place a spool on a flat surface. Make sure the twine unwinds over the top of the spool (as shown to the right). Pull the string gently so the spool rolls (doesn't slide). The spool will roll in the direction you are pulling.

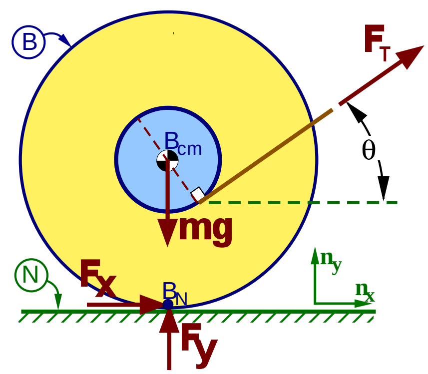
Now turn the spool over so the twine unwinds from underneath (as shown to the right). Again, pull the string gently so the spool rolls (no slipping). Which way does the spool roll (**A** or **B**)?

- **A**. In the direction you pull (spool rolls towards you).
- **B**. Opposite the way you pull (spool rolls away from you).

Analytical solution for 2D model of rolling spool

The following figure shows a rope wrapped around the axle of a wheel. The axle and wheel are rigidly connected and constitute a symmetrical rigid spool B. The wheel is in contact with a **rough** flat horizontal plane N at point B_N of B. Right-handed orthogonal units vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are fixed in N with $\widehat{\mathbf{n}}_x$ horizontally right, $\widehat{\mathbf{n}}_y$ vertically upward, and $\widehat{\mathbf{n}}_z$ parallel to B's angular velocity in N.

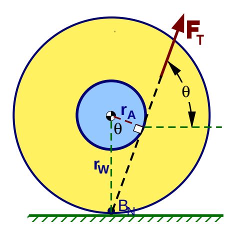
Description of scalar quantities	Symbol
Local gravitational constant	g
Mass of B	m
B's moment of inertia about its symmetry axis	I
Radius of B's axle	r_A
Radius of B's wheel	r_W
Rope's angle with the horizontal	θ
Tension in rope (regarded as non-negative)	F_T
$\widehat{\mathbf{n}}_x$ measure of force on B from N at B_N	F_x
$\widehat{\mathbf{n}}_y$ measure of force on B from N at B_N	F_y
$-\widehat{\mathbf{n}}_z$ measure of B's angular velocity in N	ω



Insights from static equilibrium of the spool

In static equilibrium, the $\widehat{\mathbf{n}}_z$ component of the moment of all forces on B about B_N is **0** (stated below), which simplifies to the following scalar equation.

$$\mathbf{M}_z^{B/B_N} \;=\; [r_A - r_W\ \cos(\theta)]\, F_T\, \widehat{\mathbf{n}}_z \;=\; \mathbf{0} \qquad \Rightarrow \qquad r_A \,-\, r_W\ \cos(\theta) \;=\; 0$$



Static equilibrium

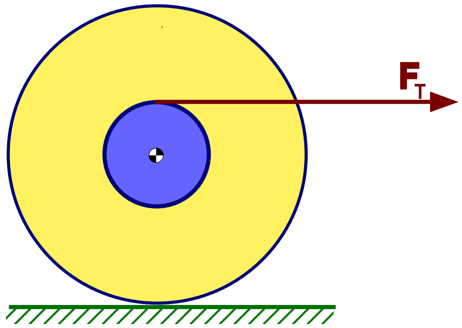
When $\mathbf{M}_z^{B/B_N} \neq \mathbf{0}$, the spool is not in static equilibrium and one may surmise:

Mathematical condition	Direction of $\mathbf{M}_z^{B/B_N}$	Implied motion	Happens when
$r_A - r_W\ \cos(\theta)\ <\ 0$	Clockwise	Spool rolls right	$0° \ \leq\ \theta\ <\ \text{acos}(\frac{r_A}{r_w})$
$r_A - r_W\ \cos(\theta)\ >\ 0$	Counterclockwise	Spool rolls left	$\text{acos}(\frac{r_A}{r_w})\ <\ \theta\ \leq\ 90°$

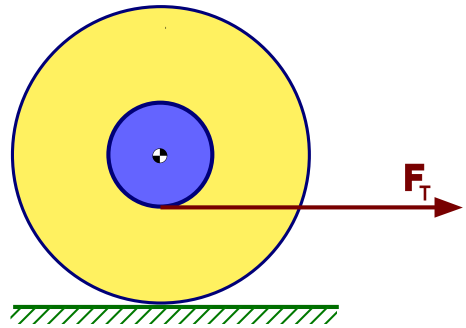
Note: The **direction** of $\mathbf{M}_z^{B/B_N}$ and roll can be determined **without calculation**.
Just use the cross-product right-hand rule to calculate the moment of the rope's force about B_N.

[4] The word "human" comes from "humus" meaning "Earth" (grounded). Humility and humble mean grounded in reality, e.g., the more you know, the more you are aware of your own ignorance.

To ensure an intelligible solution, we show the special case when the rope is horizontal ($\theta = 0$).



The wheel rolls **right.**



The wheel rolls **right!**

Using dynamics to predict rolling direction

An efficient method for doing dynamic analysis on the spool **without** calculating the unknowns F_x and F_y associated with the unknown reaction force at B_N is to employ Euler's (1736) 2D rigid body equation of motion **about point** B_N as[5]

$$\mathbf{M}_z^{B/B_N} \;=\; I_z^{B/B_N}\,\boldsymbol{\alpha} \;+\; m\;\mathbf{r}^{B_{cm}/B_N} \times \mathbf{a}^{B_N}$$

I_z^{B/B_N}	$= I \;+\; m\,r_W^2$	B's moment of inertia for line through B_N and parallel to $\widehat{\mathbf{n}}_z$
$\boldsymbol{\alpha}$	$= \text{-}\dot{\omega}\,\widehat{\mathbf{n}}_z$	B's angular acceleration in N
$\mathbf{r}^{B_{cm}/B_N}$	$= r_W\;\widehat{\mathbf{n}}_y$	Position vector of B_{cm} (B's center of mass) from B_N.
$\mathbf{a}^{B_N}$	$= r_W\,\omega^2\,\widehat{\mathbf{n}}_y$	Acceleration of B_N in N

Carrying out the vector operations, subsequent dot-product with $\widehat{\mathbf{n}}_z$, and rearrangement yields

$$[r_A - r_W\;\cos(\theta)]\,F_T \;+\; (I + m\,r_W^2)\,\dot{\omega} \qquad \Rightarrow \qquad \dot{\omega} \;=\; \frac{\text{-}[r_A - r_W\;\cos(\theta)]\,F_T}{I + m\,R^2}$$

Mathematical condition	Direction of $\dot{\omega}$	Implied motion	Happens when
$r_A - r_W\;\cos(\theta)\; < \;0$	Clockwise	Spool rolls right	$0^\circ \;\le\; \theta \;<\; \text{acos}(\frac{r_A}{r_w})$
$r_A - r_W\;\cos(\theta)\; > \;0$	Counterclockwise	Spool rolls left	$\text{acos}(\frac{r_A}{r_w}) \;<\; \theta \;\le\; 90^\circ$

This puzzle appeared Sept. 19, 2010 in Wordplay, the crossword Blog of the New York Times.
Numberplay: Which Way Will It Roll? with Gary Antonick, Apoorva Rajagopal, and Paul Mitiguy.
http://wordplay.blogs.nytimes.com/2011/09/19/numberplay-which-way-will-it-roll/

[5] Alternately, form 2 scalar equations with $\mathbf{F} = m\,\mathbf{a}$ and 1 equation with Euler's equation about B_{cm}, i.e., $\mathbf{M}_z^{B/B_{cm}} = I\,\boldsymbol{\alpha}$.