4.1 ♣ Soh Cah Toa: Sine, cosine, tangent as ratios of sides of a right triangle. (Section 1.4)

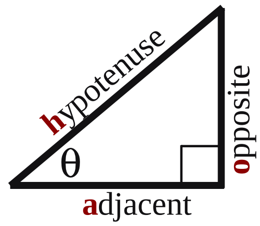
The following shows a ***right triangle*** with one of its angles labeled as θ.
Note: A ***right triangle*** is a triangle with a 90° angle.

Write definitions for sine, cosine, and tangent in terms of:

- **h**ypotenuse – the triangle's longest side (opposite the 90° angle)
- **o**pposite – the side opposite to θ
- **a**djacent – the side adjacent to θ

Note: A mnemonic for these definitions is "***Soh Cah Toa***".

I can draw a triangle with a negative-length side	**True/False**
Using the **limited** definition shown right, the sine of an angle can be negative.	**True/False**



$\sin(\theta) \triangleq \frac{\square}{\square}$

$\cos(\theta) \triangleq \frac{\square}{\square}$

$\tan(\theta) \triangleq \frac{\square}{\square}$

4.2 ♣ Pythagorean theorem and law of cosines - memorize. (Section 1.4.1).

Draw a right-triangle with a hypotenuse of length c and other sides of length a and b. Relate c to a and b with the ***Pythagorean theorem***.

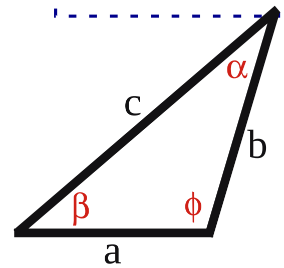
Result: $c^2 = \square + \square$

A non-right-triangle has angles α, β, ϕ opposite sides a, b, c, respectively.
Use the ***law of cosines*** to complete each formula below.

Result: $c^2 = a^2 + b^2 - 2\,a\,b\cos(\phi)$

$a^2 = \square + \square - \square$

$b^2 = \square + \square - \square$

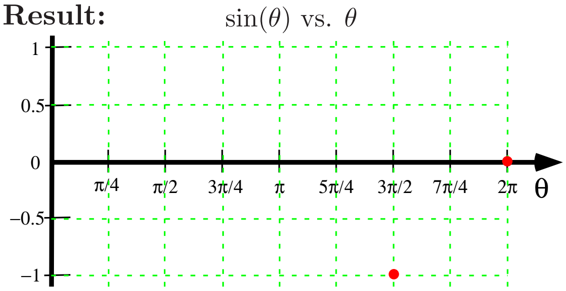


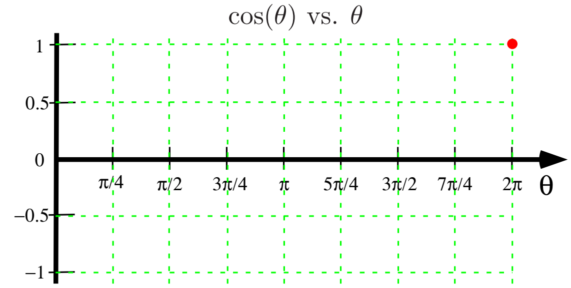
The ***Pythagorean theorem*** is a special case of the ***law of cosines***. **True/False**. (circle one).

4.3 ♣ Graphing sine and cosine - (a now-obvious invention from 1730 A.D.) (Section 1.4.2)

Graph sine and cosine as functions of the angle θ in radians over the range $0 \le \theta \le 2\pi$.
The mathematician ______ was first to regard sine and cosine as **functions** (not just ratios of sides of a triangle).

Result: $\sin(\theta)$ vs. θ $\cos(\theta)$ vs. θ





4.4 ♣ Ranges for arguments and return values for inverse trigonometric functions.

Determine all real return values and argument values for the following **real** trigonometric and inverse-trigonometric functions in computer languages such as Java, C^{++}, MATLAB®, **M**otion**G**enesis, ...

Range of return values for z	Function	Range of argument values for x	Note
___ $\le z \le$ ___	$z = \cos(x)$	___ $< x <$ ___	
___ $\le z \le$ ___	$z = \sin(x)$	___ $< x <$ ___	
$-\infty < z < \infty$	$z = \tan(x)$	$-\infty < x < \infty$	$x \ne \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
___ $\le z \le$ ___	$z = \text{acos}(x)$	___ $\le x \le$ ___	
___ $\le z \le$ ___	$z = \text{asin}(x)$	___ $\le x \le$ ___	
$-\pi/2 < z < \pi/2$	$z = \text{atan}(x)$	$-\infty < x < \infty$	

4.5 ♣ What is an angle? (Section 5.4).

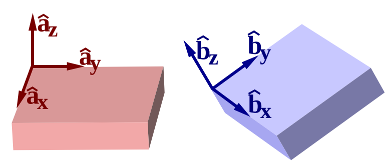
Draw the "geometry equipment" listed in the 1^{st} column of the following table.
Complete the 2^{nd} column with appropriate ranges for the angle θ (in degrees).

"Geometry equipment"	**Draw**	Appropriate range for θ
2 lines		$0° \le \theta \le$ ___
Vector and line		___ $\le \theta \le$ ___
2 vectors		___ $\le \theta \le$ ___
2 vectors and a sense of positive rotation		___ $< \theta \le$ ___
2 vectors, a sense of $\pm$ rotation, and time-history/continuity	Not applicable	___ $< \theta <$ ___

4.6 Calculating dot-products, cross-products, and angles between vectors. (Section 5.2.3).

The ${}^aR^b$ rotation table relates two sets of right-handed, orthogonal, unit vectors, namely $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$.

${}^aR^b$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$
$\hat{a}_x$	0.9623	−0.0842	0.2588
$\hat{a}_y$	0.1701	0.9284	−0.3304
$\hat{a}_z$	−0.2125	0.3619	0.9077



(a) Efficiently determine the following dot-products and angles between vectors (2^+ significant digits).
Then perform the following calculations involving $\vec{v}_1 = 2\,\hat{a}_x$ and $\vec{v}_2 = \hat{a}_x + \hat{b}_x$.

$\hat{a}_x \cdot \hat{a}_x =$ ___	$\hat{a}_y \cdot \hat{a}_z =$ ___	$\hat{b}_z \cdot \hat{b}_y =$ ___
$\hat{a}_x \cdot \hat{b}_x =$ ___	$\hat{a}_x \cdot \hat{b}_y =$ ___	$\hat{b}_z \cdot \hat{a}_y =$ ___
$\angle(\hat{a}_y, \hat{a}_y) =$ ___ °	$\angle(\hat{b}_z, \hat{b}_x) =$ ___ °	
$\angle(\hat{a}_y, \hat{b}_y) =$ ___ °	$\angle(\hat{b}_y, \hat{a}_z) =$ ___ °	

Result: $\vec{v}_1 \cdot \vec{v}_2 =$ ___ $\qquad \angle(\vec{v}_1, \vec{v}_2) =$ ___ °

$\vec{v}_1 \times \vec{v}_2 =$ ___ $\hat{b}_y +$ ___ $\hat{b}_z =$ ___ $\hat{a}_y +$ ___ $\hat{a}_z$

(b) Express the unit vector $\hat{u}$ in the direction of $3\,\hat{a}_z + 4\,\hat{b}_z$ in terms of $\hat{a}_z$ and $\hat{b}_z$.
Express $\vec{v} = \hat{a}_y + \hat{b}_y$ in terms of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$.
Result: $\hat{u} =$ ___ $\hat{a}_z +$ ___ $\hat{b}_z$

$\vec{v} =$ ___ $\hat{a}_x +$ ___ $\hat{a}_y +$ ___ $\hat{a}_z$

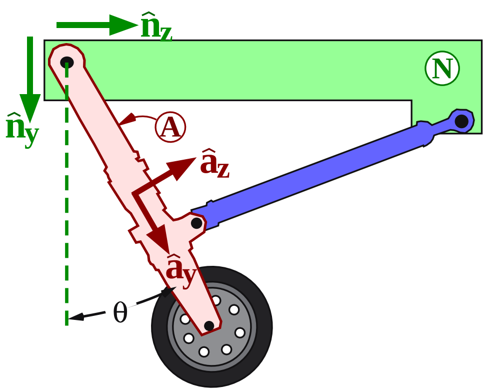
4.7 ♣ Efficient calculation of the inverse of a rotation matrix. (Section 5.2.2).

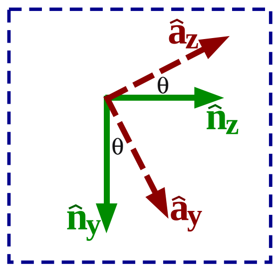
The following rotation matrix R relates two right-handed, orthogonal, unitary bases.
Calculate its inverse by-hand (no calculator) in less than 30 seconds.

$$R \;=\; \begin{bmatrix} 0.3830 & -0.6634 & 0.6428 \\ 0.9237 & 0.2795 & -0.2620 \\ -0.0058 & 0.6941 & 0.7198 \end{bmatrix} \qquad \Rightarrow \qquad R^{-1} \;=\; \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

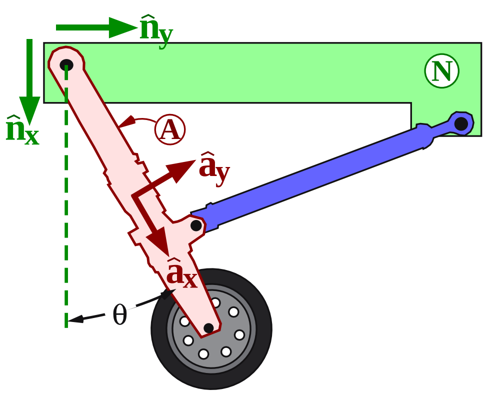
4.8 ♣ SohCahToa: Rotation tables for a landing gear system. (Section 5.3).

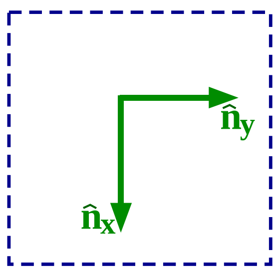
The figures below show three versions of the same landing gear system with a strut A that has a simple rotation relative to a fuselage N. Each figure has a different orientation for right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ (fixed in N) and $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ (fixed in A). **Redraw** $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ so it is **easy to see** sines and cosines. Then, form the ${}^aR^n$ rotation table for each figure.[1]



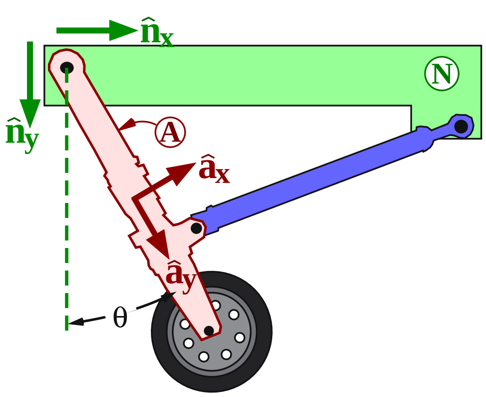


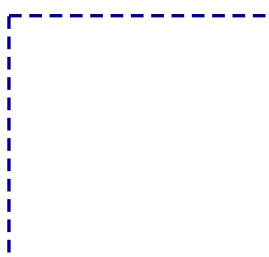
${}^aR^n$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$	$\hat{\mathbf{n}}_z$
$\hat{\mathbf{a}}_x$	1		
$\hat{\mathbf{a}}_y$	0	$\cos(\theta)$	$\sin(\theta)$
$\hat{\mathbf{a}}_z$	0	$-\sin(\theta)$	$\cos(\theta)$





${}^aR^n$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$	$\hat{\mathbf{n}}_z$
$\hat{\mathbf{a}}_x$			
$\hat{\mathbf{a}}_y$			
$\hat{\mathbf{a}}_z$			





${}^aR^n$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$	$\hat{\mathbf{n}}_z$
$\hat{\mathbf{a}}_x$			
$\hat{\mathbf{a}}_y$			
$\hat{\mathbf{a}}_z$			

[1] Each figure has two missing vectors (e.g., $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{a}}_x$ are missing from the first figure). Use the fact that each set of vectors is **right-handed** to add the missing vectors to each figure.

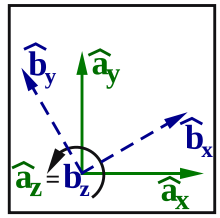
4.9 ♣ Rotation table concepts: What is an angle. (Section 5.4)

Given: Two sets of right-handed, orthogonal, unitary bases $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.

Question: Determine a numerical expression for each element of the 3×3 rotation table ${}^bR^a$ below so $\widehat{\mathbf{b}}_z = \widehat{\mathbf{a}}_z$ and the angle between $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{a}}_x$ is 30°.
Draw $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$, clearly showing the relative orientation of the two bases.

Question: Is ${}^bR^a$ unique when $\widehat{\mathbf{b}}_z = \widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{b}}_x \cdot \widehat{\mathbf{a}}_x = \frac{\sqrt{3}}{2}$? **Yes/No**.
If your answer is **No**, **draw** an alternative orientation for $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.

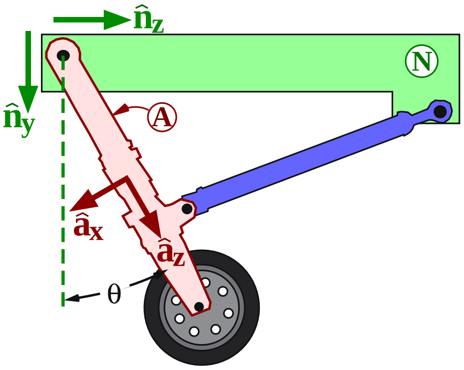
${}^bR^a$	$\widehat{\mathbf{a}}_x$	$\widehat{\mathbf{a}}_y$	$\widehat{\mathbf{a}}_z$
$\widehat{\mathbf{b}}_x$			
$\widehat{\mathbf{b}}_y$			
$\widehat{\mathbf{b}}_z$			



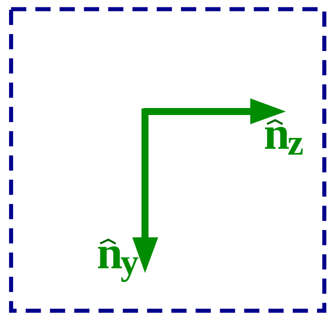
If "no", alternative

4.10 ♣ SohCahToa: Rotation table with disorderly unit vectors. (Section 5.3)

The following figure shows a landing gear system with a strut A that has a simple rotation relative to a fuselage N. Right-handed sets of orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are fixed in A and N, respectively. θ is the angle from $\widehat{\mathbf{n}}_y$ to $\widehat{\mathbf{a}}_z$ with $^+\widehat{\mathbf{n}}_x$ sense.



Redraw $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ in a geometrically suggestive way for forming the ${}^aR^n$ rotation matrix with sine and cosine.



Note: When $\theta = 0$, $\widehat{\mathbf{a}}_x \neq \widehat{\mathbf{n}}_x$ and $\widehat{\mathbf{a}}_y \neq \widehat{\mathbf{n}}_y$ and $\widehat{\mathbf{a}}_z \neq \widehat{\mathbf{n}}_z$.
Thus, $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ are "**disordered**" with $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.

Complete the blanks in the equations relating $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ to $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ and in the ${}^aR^n$ rotation table.

$$\widehat{\mathbf{a}}_x = \underline{\qquad}\,\widehat{\mathbf{n}}_x + \underline{\qquad}\,\widehat{\mathbf{n}}_y + \underline{\qquad}\,\widehat{\mathbf{n}}_z$$
$$\widehat{\mathbf{a}}_y = \underline{\qquad}\,\widehat{\mathbf{n}}_x + \underline{\qquad}\,\widehat{\mathbf{n}}_y + \underline{\qquad}\,\widehat{\mathbf{n}}_z$$
$$\widehat{\mathbf{a}}_z = \underline{\qquad}\,\widehat{\mathbf{n}}_x + \underline{\qquad}\,\widehat{\mathbf{n}}_y + \underline{\qquad}\,\widehat{\mathbf{n}}_z$$

${}^aR^n$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{a}}_x$			
$\widehat{\mathbf{a}}_y$			
$\widehat{\mathbf{a}}_z$			