

4.1 ♣ **Soh Cah Toa: Sine, cosine, tangent as ratios of sides of a right triangle.** (Section 1.4)

The following shows a *right triangle* with one of its angles labeled as θ .

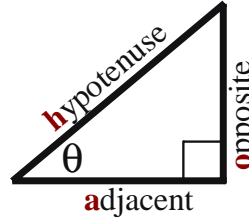
Note: A *right triangle* is a triangle with a 90° angle.

Write definitions for sine, cosine, and tangent in terms of:

- **h**ypotenuse – the triangle’s longest side (opposite the 90° angle)
- **o**pposite – the side opposite to θ
- **a**djacent – the side adjacent to θ

Note: A mnemonic for these definitions is “Soh Cah Toa”.

I can draw a triangle with a negative-length side True/False
 Using the limited definition shown right, True/False
 the sine of an angle can be negative.



$\sin(\theta) \triangleq \frac{\text{opposite}}{\text{hypotenuse}}$

$\cos(\theta) \triangleq \frac{\text{adjacent}}{\text{hypotenuse}}$

$\tan(\theta) \triangleq \frac{\text{opposite}}{\text{adjacent}}$

4.2 ♣ **Pythagorean theorem and law of cosines - memorize.** (Section 1.4.1).

Draw a right-triangle with a hypotenuse of length c and other sides of length a and b . Relate c to a and b with the *Pythagorean theorem*.

Result:

$$c^2 = \square + \square$$

A non-right-triangle has angles α, β, ϕ opposite sides a, b, c , respectively.

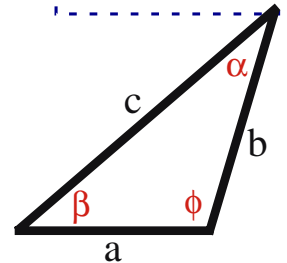
Use the *law of cosines* to complete each formula below.

Result:

$$c^2 = a^2 + b^2 - 2ab \cos(\phi)$$

$$a^2 = \square + \square - \square$$

$$b^2 = \square + \square - \square$$



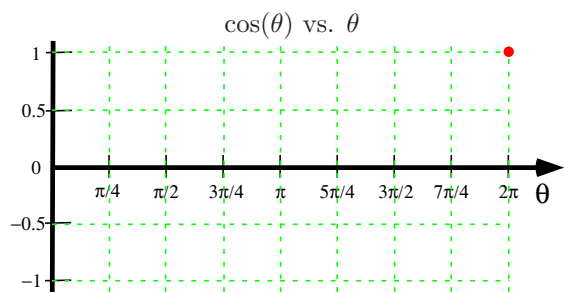
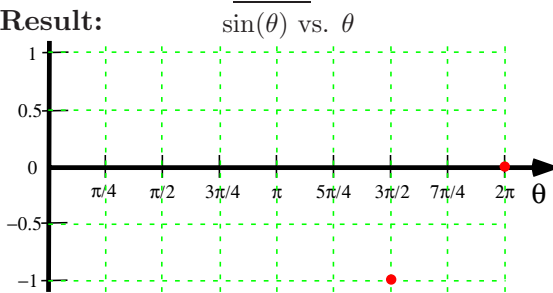
The *Pythagorean theorem* is a special case of the *law of cosines*. True/False. (circle one).

4.3 ♣ **Graphing sine and cosine - (a now-obvious invention from 1730 A.D.)** (Section 1.4.2)

Graph sine and cosine as functions of the angle θ in radians over the range $0 \leq \theta \leq 2\pi$.

The mathematician was first to regard sine and cosine as functions (not just ratios of sides of a triangle).

Result:



4.4 ♣ Ranges for arguments and return values for inverse trigonometric functions.

Determine all real return values and argument values for the following **real** trigonometric and inverse-trigonometric functions in computer languages such as Java, C++, MATLAB®, MotionGenesis, ...

Range of return values for z	Function	Range of argument values for x	Note
<input type="text"/> $\leq z \leq$ <input type="text"/>	$z = \cos(x)$	<input type="text"/> $< x <$ <input type="text"/>	
<input type="text"/> $\leq z \leq$ <input type="text"/>	$z = \sin(x)$	<input type="text"/> $< x <$ <input type="text"/>	
<input type="text"/> $< z <$ <input type="text"/>	$z = \tan(x)$	<input type="text"/> $< x <$ <input type="text"/>	$x \neq \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$
<input type="text"/> $\leq z \leq$ <input type="text"/>	$z = \text{acos}(x)$	<input type="text"/> $\leq x \leq$ <input type="text"/>	
<input type="text"/> $\leq z \leq$ <input type="text"/>	$z = \text{asin}(x)$	<input type="text"/> $\leq x \leq$ <input type="text"/>	
<input type="text"/> $< z <$ <input type="text"/>	$z = \text{atan}(x)$	<input type="text"/> $< x <$ <input type="text"/>	
<input type="text"/> $\leq z \leq$ <input type="text"/>	$z = \text{atan2}(y, x)$	<input type="text"/> $< y <$ <input type="text"/> <input type="text"/> $< x <$ <input type="text"/>	$\text{atan2}(0, 0)$ is undefined

4.5 ♣ What is an angle? (Section 5.6).

Draw the “geometry equipment” listed in the 1st column of the following table. Complete the 2nd column with appropriate ranges for the angle θ (in degrees).

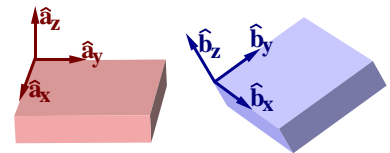


“Geometry equipment”	Draw	Appropriate range for θ
2 lines	<input type="text"/>	$0^\circ \leq \theta \leq$ <input type="text"/>
Vector and line	<input type="text"/>	<input type="text"/> $\leq \theta \leq$ <input type="text"/>
2 vectors	<input type="text"/>	<input type="text"/> $\leq \theta \leq$ <input type="text"/>
2 vectors and a sense of positive rotation	<input type="text"/>	<input type="text"/> $< \theta \leq$ <input type="text"/>
2 vectors, a sense of \pm rotation, and time-history/continuity	Not applicable	<input type="text"/> $< \theta <$ <input type="text"/>

4.6 Calculating dot-products, cross-products, and angles between vectors. (Section 5.4.3).

The ${}^aR^b$ rotation table relates two sets of right-handed, orthogonal, unit vectors, namely $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$ and $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$.

${}^aR^b$	$\hat{\mathbf{b}}_x$	$\hat{\mathbf{b}}_y$	$\hat{\mathbf{b}}_z$
$\hat{\mathbf{a}}_x$	0.9623	-0.0842	0.2588
$\hat{\mathbf{a}}_y$	0.1701	0.9284	-0.3304
$\hat{\mathbf{a}}_z$	-0.2125	0.3619	0.9077



- (a) Efficiently determine the following dot-products and angles between vectors (2⁺ significant digits). Then perform the following calculations involving $\vec{\mathbf{v}}_1 = 2\hat{\mathbf{a}}_x$ and $\vec{\mathbf{v}}_2 = \hat{\mathbf{a}}_x + \hat{\mathbf{b}}_x$.

$\hat{\mathbf{a}}_x \cdot \hat{\mathbf{a}}_x =$ <input type="text"/>	$\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_z =$ <input type="text"/>	$\hat{\mathbf{b}}_z \cdot \hat{\mathbf{b}}_y =$ <input type="text"/>
$\hat{\mathbf{a}}_x \cdot \hat{\mathbf{b}}_x =$ <input type="text"/>	$\hat{\mathbf{a}}_x \cdot \hat{\mathbf{b}}_y =$ <input type="text"/>	$\hat{\mathbf{b}}_z \cdot \hat{\mathbf{a}}_y =$ <input type="text"/>
$\angle(\hat{\mathbf{a}}_y, \hat{\mathbf{a}}_y) =$ <input type="text"/> °	$\angle(\hat{\mathbf{b}}_z, \hat{\mathbf{b}}_x) =$ <input type="text"/> °	
$\angle(\hat{\mathbf{a}}_y, \hat{\mathbf{b}}_y) =$ <input type="text"/> °	$\angle(\hat{\mathbf{b}}_y, \hat{\mathbf{a}}_z) =$ <input type="text"/> °	

Result: $\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 =$ $\angle(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2) =$ °

$\vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2 =$ $\hat{\mathbf{b}}_y +$ $\hat{\mathbf{b}}_z =$ $\hat{\mathbf{a}}_y +$ $\hat{\mathbf{a}}_z$

- (b) Express the unit vector $\hat{\mathbf{u}}$ in the direction of $3\hat{\mathbf{a}}_z + 4\hat{\mathbf{b}}_z$ in terms of $\hat{\mathbf{a}}_z$ and $\hat{\mathbf{b}}_z$. Express $\vec{\mathbf{v}} = \hat{\mathbf{a}}_y + \hat{\mathbf{b}}_y$ in terms of $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$.

Result: $\hat{\mathbf{u}} =$ $\hat{\mathbf{a}}_z +$ $\hat{\mathbf{b}}_z$

$\vec{\mathbf{v}} =$ $\hat{\mathbf{a}}_x +$ $\hat{\mathbf{a}}_y +$ $\hat{\mathbf{a}}_z$

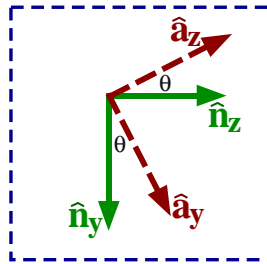
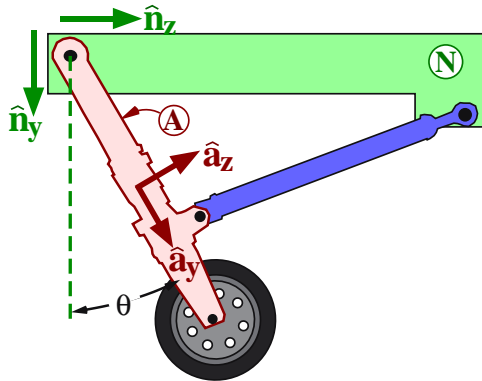
4.7 ♣ **Efficient calculation of the inverse of a rotation matrix.** (Section 5.4.2).

The following rotation matrix R relates two right-handed, orthogonal, unitary bases. Calculate its inverse by-hand (no calculator) in less than 30 seconds.

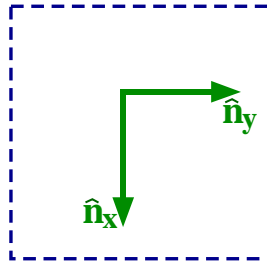
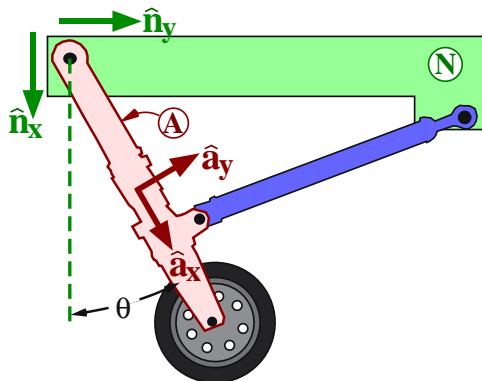
$$R = \begin{bmatrix} 0.3830 & -0.6634 & 0.6428 \\ 0.9237 & 0.2795 & -0.2620 \\ -0.0058 & 0.6941 & 0.7198 \end{bmatrix} \Rightarrow R^{-1} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

4.8 ♣ **SohCahToa: Rotation tables for a landing gear system.** (Section 5.5).

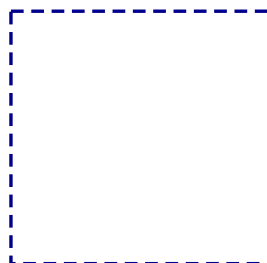
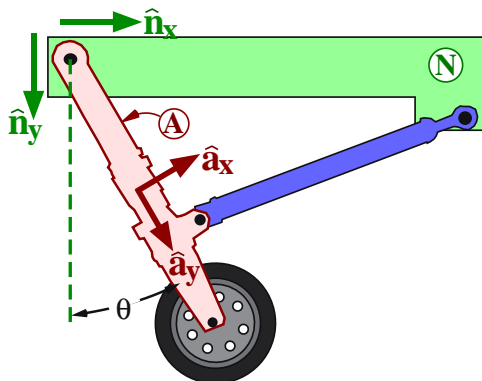
The figures below show three versions of the same landing gear system with a strut A that has a simple rotation relative to a fuselage N . Each figure has a different orientation for right-handed orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$ (fixed in N) and $\hat{a}_x, \hat{a}_y, \hat{a}_z$ (fixed in A). **Redraw** $\hat{n}_x, \hat{n}_y, \hat{n}_z$ and $\hat{a}_x, \hat{a}_y, \hat{a}_z$ so it is **easy to see** sines and cosines. Then, form the ${}^aR^n$ rotation table for each figure.¹



${}^aR^n$	\hat{n}_x	\hat{n}_y	\hat{n}_z
\hat{a}_x	1	<input type="text"/>	<input type="text"/>
\hat{a}_y	0	$\cos(\theta)$	$\sin(\theta)$
\hat{a}_z	0	$-\sin(\theta)$	$\cos(\theta)$



${}^aR^n$	\hat{n}_x	\hat{n}_y	\hat{n}_z
\hat{a}_x	<input type="text"/>	<input type="text"/>	<input type="text"/>
\hat{a}_y	<input type="text"/>	<input type="text"/>	<input type="text"/>
\hat{a}_z	<input type="text"/>	<input type="text"/>	<input type="text"/>



${}^aR^n$	\hat{n}_x	\hat{n}_y	\hat{n}_z
\hat{a}_x	<input type="text"/>	<input type="text"/>	<input type="text"/>
\hat{a}_y	<input type="text"/>	<input type="text"/>	<input type="text"/>
\hat{a}_z	<input type="text"/>	<input type="text"/>	<input type="text"/>

¹Each figure has two missing vectors (e.g., \hat{n}_x and \hat{a}_x are missing from the first figure). Use the fact that each set of vectors is **right-handed** to add the missing vectors to each figure.

4.9 ♣ **Rotation table concepts: What is an angle.** (Section 5.6)

Given: Two sets of right-handed, orthogonal, unitary bases $\hat{a}_x, \hat{a}_y, \hat{a}_z$ and $\hat{b}_x, \hat{b}_y, \hat{b}_z$.

Question: Determine a numerical expression for each element of the 3×3 rotation table ${}^bR^a$ below so $\hat{b}_z = \hat{a}_z$ and the angle between \hat{b}_x and \hat{a}_x is 30° .

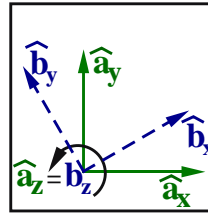
Draw $\hat{b}_x, \hat{b}_y, \hat{b}_z$, clearly showing the relative orientation of the two bases.

Question: Are there other orientations of $\hat{b}_x, \hat{b}_y, \hat{b}_z$ and $\hat{a}_x, \hat{a}_y, \hat{a}_z$ such that $\hat{b}_z = \hat{a}_z$ and the angle between \hat{b}_x and \hat{a}_x is 30° ? **Yes/No.**

Question: Is ${}^bR^a$ unique when $\hat{b}_z = \hat{a}_z$ and $\hat{b}_x \cdot \hat{a}_x = \frac{\sqrt{3}}{2}$? **Yes/No.**

If your answer is **No**, **draw** an alternative orientation for $\hat{b}_x, \hat{b}_y, \hat{b}_z$.

${}^bR^a$	\hat{a}_x	\hat{a}_y	\hat{a}_z
\hat{b}_x			
\hat{b}_y			
\hat{b}_z			

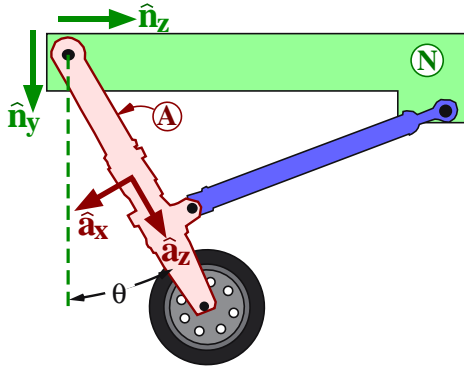


If "no", alternative

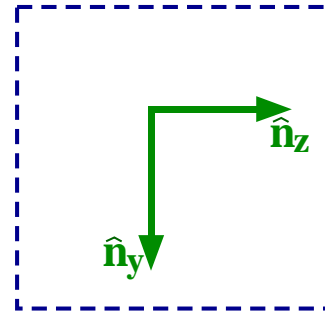


4.10 ♣ **SohCahToa: Rotation table with disorderly unit vectors.** (Section 5.5)

The following figure shows a landing gear system with a strut A that has a simple rotation relative to a fuselage N. Right-handed sets of orthogonal unit vectors $\hat{a}_x, \hat{a}_y, \hat{a}_z$ and $\hat{n}_x, \hat{n}_y, \hat{n}_z$ are fixed in A and N, respectively. θ is the angle from \hat{n}_y to \hat{a}_z with $+\hat{n}_x$ sense.



Redraw $\hat{a}_x, \hat{a}_y, \hat{a}_z$ in a geometrically suggestive way for forming the ${}^aR^n$ rotation matrix with sine and cosine.



Note: When $\theta = 0$, $\hat{a}_x \neq \hat{n}_x$ and $\hat{a}_y \neq \hat{n}_y$ and $\hat{a}_z \neq \hat{n}_z$. Thus, $\hat{a}_x, \hat{a}_y, \hat{a}_z$ are "**disordered**" with $\hat{n}_x, \hat{n}_y, \hat{n}_z$.

Complete the blanks in the equations relating $\hat{a}_x, \hat{a}_y, \hat{a}_z$ to $\hat{n}_x, \hat{n}_y, \hat{n}_z$ and in the ${}^aR^n$ rotation table.

$$\hat{a}_x = \underline{\hspace{1cm}} \hat{n}_x + \underline{\hspace{1cm}} \hat{n}_y + \underline{\hspace{1cm}} \hat{n}_z$$

$$\hat{a}_y = \underline{\hspace{1cm}} \hat{n}_x + \underline{\hspace{1cm}} \hat{n}_y + \underline{\hspace{1cm}} \hat{n}_z$$

$$\hat{a}_z = \underline{\hspace{1cm}} \hat{n}_x + \underline{\hspace{1cm}} \hat{n}_y + \underline{\hspace{1cm}} \hat{n}_z$$

${}^aR^n$	\hat{n}_x	\hat{n}_y	\hat{n}_z
\hat{a}_x			
\hat{a}_y			
\hat{a}_z			