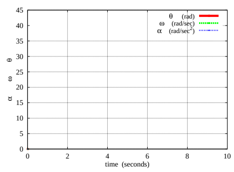
6.1 ♣ FE/EIT Review – Motion graph: $\mathbf{T} \underset{(2D)}{=} \mathbf{I}\,\alpha \qquad \alpha = \frac{d\omega}{dt} \qquad \omega \underset{(2D)}{=} \frac{d\theta}{dt}$

A wind turbine generates electricity from a torque (created by wind forces) modeled as $\mathrm{T} = 20\,\frac{\mathrm{N\,m}}{\mathrm{sec}} * t$. Equations relating the wind turbine's angular acceleration α, angular velocity ω, and angle θ are shown above, where moment of inertia $\mathrm{I} = 80\ \mathrm{kg\,m^2}$. Initially, (at $t = 0$), $\omega = 0$ and $\theta = 0$.

Determine $\alpha(t)$, $\omega(t)$, $\theta(t)$.

Result:

$\alpha(t) = \frac{\square}{\square} \qquad \omega(t) = \frac{\square}{\square} \qquad \theta(t) = \frac{\square}{\square}$



Optional: Graph $\alpha(t)$, $\omega(t)$, and $\theta(t)$ for $0 \le t \le 10$ sec.

6.2 ♣ Draw a rigid body, center of mass, vector basis, and rigid frame. (Section 7.2.3)

Draw a ***rigid body*** B, shaped like a uniform-density doughnut (with a hole).

Draw a properly located ***center of mass*** symbol ◐ and label this point B_{cm}.

Draw a right-handed orthogonal ***basis*** fixed in B with unit vectors $\widehat{\mathbf{b}}_1$, $\widehat{\mathbf{b}}_2$, $\widehat{\mathbf{b}}_3$.

Draw a ***rigid frame*** with ***origin point*** B_o (at a location different than B_{cm}) and a right-handed orthogonal ***basis*** fixed in B with unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.

6.3 ♣ Words and pictures for ${}^bR^a$, ${}^N\vec{\boldsymbol{\omega}}^B$, ${}^N\vec{\boldsymbol{\alpha}}^B$. (Chapters 5 and 7)

Complete each blank below with a phrase to the right.

3D orthonormal basis	rigid body	reference frame
rotation matrix	angular velocity	angular acceleration

${}^aR^b$ R denotes ______. a is a ______. b is a ______.	${}^N\vec{\boldsymbol{\omega}}^B$ $\vec{\boldsymbol{\omega}}$ denotes ______. ${}^N\vec{\boldsymbol{\alpha}}^B$ $\vec{\boldsymbol{\alpha}}$ denotes ______. N is a ______ or ______ or ______. B is a ______ or ______ or ______.
Draw a and b.	**Draw** N and B.

6.4 ♣ Definition of angular velocity? (Section 7.3.3).

A "popular" definition of angular velocity is $\vec{\boldsymbol{\omega}} \triangleq \dot{\theta}\,\vec{\mathbf{k}}$. This definition is generally useful for calculating angular velocity and proving its properties (for both 2D and/or 3D analysis). **True/False**

6.5 ♣ Optional: Textbook/Internet definition of 3D angular velocity (Section 7.3).

Renowned dynamicist Thomas Kane said angular velocity is one of the "**most misunderstood concepts in kinematics**." Report a definition of angular velocity and determine if the quantities appearing in the definition are **rigorously defined** - and whether the definition is generally applicable for 3D kinematics or only applies for simple angular velocity (e.g., $\vec{\boldsymbol{\omega}} = \dot{\theta}\,\vec{\mathbf{k}}$ described in Section 7.3.3).

Note: A definition should be able to **prove** important theorems [such as the angular velocity addition theorem of equation (7.4) and the golden rule for vector differentiation in equation (7.1)] and allow for angular velocity **calculations**.

Source (reference). List textbook or .html link	Definition. Report the defining equation/property.	Rigorously defined?	Works for 3D kinematics?
		Yes/No	**Yes/No**

6.6 ♣ Angular velocity of a Ferris-wheel seat (courtesy of David Levinson).

Known: The rigid seat on a Ferris wheel does not change its orientation relative to ground as the Ferris wheel rotates.
Decide: The seat's angular velocity $\vec{\boldsymbol{\omega}}$ relative to ground (circle one):

Is zero	Is constant	Varies	Does not exist
$\vec{\boldsymbol{\omega}} = \vec{\mathbf{0}}$	$\vec{\boldsymbol{\omega}} = \vec{\textbf{Constant}} \neq \vec{\mathbf{0}}$	$\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\omega}}(t)$	

6.7 ♣ Concept: What objects have a unique angular velocity/acceleration? (Sections 7.3, 7.4).
The angular velocity $\vec{\boldsymbol{\omega}}$ of some object S relative to Earth is to be determined.
This object S could be a (circle ***all*** objects that have an ***unambiguously*** defined angular velocity $\vec{\boldsymbol{\omega}}$):

Real number	Matrix	Set of points	Mass center of a rigid body
Vector	Point	Reference frame	Flexible body
3D orthogonal unit basis	Particle	3D rigid body	System of particles and bodies

Repeat for the angular acceleration $\vec{\boldsymbol{\alpha}}$ of some object S relative to Earth [box appropriate objects].

6.8 ♣ Concepts: What objects have a uniquely-defined angular velocity? (Section 7.3).

a.	The angular velocity of a	point	in a	reference frame	is well defined.	**True/False**
b.	The angular velocity of a	3D rigid body	in a	particle	is well defined.	**True/False**
c.	The angular velocity of a	3D rigid body	in a	reference frame	is well defined.	**True/False**
d.	The angular velocity of a	reference frame	in a	3D rigid body	is well defined.	**True/False**
d.	The angular velocity of a	3D rigid basis	in a	reference frame	is well defined.	**True/False**
e.	The angular velocity of a	reference frame	in a	flexible body	is well defined.	**True/False**
f.	The angular velocity of a	flexible body	in a	reference frame	is well defined.	**True/False**

6.9 ♣ Memorize the "golden rule for vector differentiation". (Section 7.3).
The angular velocity ${}^A\vec{\boldsymbol{\omega}}^B$ characterizes the time-rate of change of orientation of a reference frame (or rigid vector basis) B in a reference frame (or rigid vector basis) A.

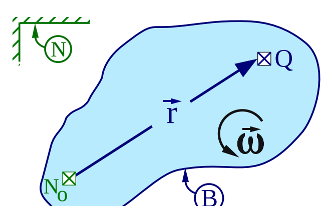
Complete the defining property of ${}^A\vec{\boldsymbol{\omega}}^B$ shown right which relates time-derivatives of **any vector** $\vec{\mathbf{v}}$ in A and B.

★ **Very important formula** ★

$$\frac{{}^A d\vec{\mathbf{v}}}{dt} \underset{(7.1)}{=} \frac{{}^B d\vec{\mathbf{v}}}{dt} + \underline{\qquad} \times \underline{\quad}$$

6.10 ♣ Vector differentiation concepts "$v = \omega\, r$". (Section 7.3).
Point Q is fixed on a rigid body B. Point N_o is fixed in a reference frame N and does not move on B. Complete the following proof that shows how $\vec{\mathbf{v}}$ (Q's velocity in N) can be written in terms of ${}^N\vec{\boldsymbol{\omega}}^B$ (B's angular velocity in N) and $\vec{\mathbf{r}}$ (Q's position from N_o).

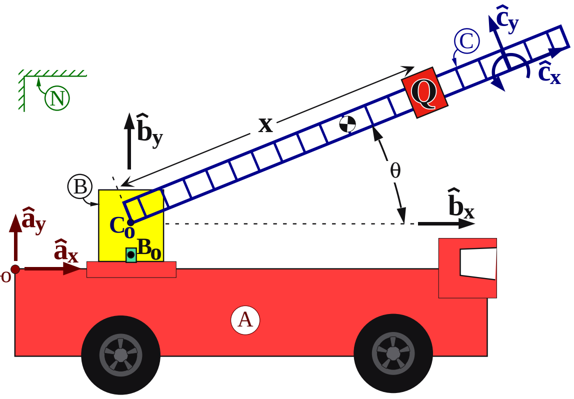
Mathematical statement	Reasoning (explain each step below with a brief phrase)
$\vec{\mathbf{v}} \triangleq \frac{{}^N d\vec{\mathbf{r}}}{dt}$	Definition of Q's velocity in N
$= \underline{\quad} + \underline{\qquad}$	
$= \vec{\mathbf{0}} + {}^N\vec{\boldsymbol{\omega}}^B \times \vec{\mathbf{r}}$	



6.11 ♣ Rotational kinematics of a fire ladder. (Sections 7.3.3, 7.3.5, 7.3.6).
The following figure shows a fire truck chassis A traveling at constant speed in straight-line motion on Earth (A does not rotate relative to Earth). Earth is a ***Newtonian reference frame*** N.
A rigid hub B is connected to fire truck A by a revolute motor at point B_o of B.
A rigid ladder C is connected to hub B by a revolute motor at point C_o of C.
A fire-fighter Q (modeled as a particle of mass m) climbs ladder C.

Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$; $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$; $\hat{\mathbf{c}}_x, \hat{\mathbf{c}}_y, \hat{\mathbf{c}}_z$; are fixed in A, B, C, with:

- $\hat{\mathbf{a}}_x$ pointing forward on the fire truck
- $\hat{\mathbf{a}}_y$ vertically-upward and from B_o to C_o
- $\hat{\mathbf{b}}_y = \hat{\mathbf{a}}_y$ parallel to the axis of the revolute motor that connects B and A
- $\hat{\mathbf{b}}_z = \hat{\mathbf{c}}_z$ parallel to the axis of the revolute motor that connects B and C
- $\hat{\mathbf{c}}_x$ directed from C_o to Q (along the ladder)



Note: **Visualize** C's "**Body yz**" (or "**Space zy**") rotation sequence in N (e.g., with a ruler).

Quantity	Symbol	Type
$\hat{\mathbf{b}}_y$ measure of B's angular velocity in A	ω_B	Constant
Angle from $\hat{\mathbf{b}}_x$ to $\hat{\mathbf{c}}_x$ with $+\hat{\mathbf{b}}_z$ sense	θ	Variable

${}^cR^b$			

(a) Complete the previous ${}^cR^b$ rotation table (to the right).
Note: ${}^cR^b$ is unnecessary for the remainder of this problem.

(b) Clarify the process to determine C's angular velocity in B and ${}^A\vec{\boldsymbol{\omega}}^B$. (Section 7.3.3).

- ${}^B\vec{\boldsymbol{\omega}}^C$ is a **simple** angular velocity because $\hat{\mathbf{b}}_z$ is a vector fixed in **both** ___ and ___.
- ${}^B\vec{\boldsymbol{\omega}}^C =$ ___. The sign (±) is determined using the ___-hand rule (sweep from ___ to ___).
- ${}^A\vec{\boldsymbol{\omega}}^B = \omega_B\,\hat{\mathbf{b}}_y$ is a **simple** angular velocity because $\hat{\mathbf{b}}_y$ is fixed in **both** ___ and ___.

(c) Form C's angular velocity in N and express it in terms of $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$. (Section 7.3.5).
Result: $${}^N\vec{\boldsymbol{\omega}}^C \underset{(7.4)}{=} {}^{\_\_}\vec{\boldsymbol{\omega}}^{\_\_} + {}^{\_\_}\vec{\boldsymbol{\omega}}^{\_\_} + {}^{\_\_}\vec{\boldsymbol{\omega}}^{\_\_} = \vec{0} + \_\_\,\hat{\mathbf{b}}_y + \_\_\,\hat{\mathbf{b}}_z$$

(d) When both ω_B and $\dot{\theta}$ are ***constant***, ${}^N\vec{\boldsymbol{\alpha}}^C = \vec{0}$. **True/False**.

(e) Write the definition for C's angular acceleration in N and form ${}^N\vec{\boldsymbol{\alpha}}^C$. (Sections 7.4, 7.3).
Result: $${}^N\vec{\boldsymbol{\alpha}}^C \underset{(7.5)}{\triangleq} \_\_\_ \qquad {}^N\vec{\boldsymbol{\alpha}}^C \underset{(7.1)}{=} \omega_B\,\dot{\theta}\,\hat{\mathbf{b}}_x + \ddot{\theta}\,\hat{\mathbf{b}}_z$$

6.12 ♣ Theorems: Rotation matrices R, angular velocity $\vec{\boldsymbol{\omega}}$, angular acceleration $\vec{\boldsymbol{\alpha}}$? (Section 7.4).

Determine the validity of each theorem to the right for general 3D motion of rigid frames A, B, C, D.

Note: Rigid basis a contains right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$ fixed in rigid frame A.
Similarly for rigid bases b, c, d fixed in frames B, C, D.

Theorem	True or false?
${}^aR^d = {}^aR^b * {}^bR^c * {}^cR^d$	**True/False**
${}^A\vec{\boldsymbol{\omega}}^D = {}^A\vec{\boldsymbol{\omega}}^B + {}^B\vec{\boldsymbol{\omega}}^C + {}^C\vec{\boldsymbol{\omega}}^D$	**True/False**
${}^A\vec{\boldsymbol{\alpha}}^D = {}^A\vec{\boldsymbol{\alpha}}^B + {}^B\vec{\boldsymbol{\alpha}}^C + {}^C\vec{\boldsymbol{\alpha}}^D$	**True/False**

6.13 Alternate formula for angular acceleration. (Section 7.3).

Show/prove ${}^N\vec{\boldsymbol{\alpha}}^B \triangleq \dfrac{{}^N d\ {}^N\vec{\boldsymbol{\omega}}^B}{dt}$ can also be calculated as ${}^N\vec{\boldsymbol{\alpha}}^B = \dfrac{{}^B d\ {}^N\vec{\boldsymbol{\omega}}^B}{dt}$.

6.14 Optional: Angular acceleration addition theorem. (Sections 7.3, 7.3.5, 7.4).

Use the angular velocity addition theorem and the definition of angular acceleration to prove:

$${}^N\vec{\boldsymbol{\alpha}}^B = {}^N\vec{\boldsymbol{\alpha}}^A + {}^A\vec{\boldsymbol{\alpha}}^B + {}^N\vec{\boldsymbol{\omega}}^A \times {}^A\vec{\boldsymbol{\omega}}^B$$