

Show work – except for ♣ fill-in-blanks-problems.

Vector functions and vector differentiation

5.1 ♣ Notations for derivatives. (Section 1.7.1).

Date	Person	Symbols for 1 st , 2 nd , and 3 rd derivatives		
1675	<input type="text"/>	$\frac{dy}{dt}$	$\frac{d^2y}{dt^2}$	$\frac{d^3y}{dt^3}$
1675	<input type="text"/>	\dot{y}	\ddot{y}	\dddot{y}
1797	<input type="text"/> (trained by Euler)	y'	y''	y'''
1850	Cauchy/Weierstrauss	$\lim_{h \rightarrow 0} \frac{y(t+h) - y(t)}{h}$?	?
1786	Legendre (introduced partials then abandoned)	$\frac{\partial y}{\partial x}$	$\frac{\partial^2 y}{\partial x^2}$	$\frac{\partial^3 y}{\partial x^3}$
1841	Jacobi (re-introduced partials again)			

There was bitter rivalry between Newton and Leibniz, and the notations of Leibniz and Newton are not entangled.

For example, $\frac{dy}{dt}$ is written in Leibniz's notation as or Newton's as .

5.2 ♣ Leibniz's shorthand notation for 3rd derivatives. (Section 1.7.1).

Write the explicit expression for the following 3rd derivative (so it contains three 1st derivatives).

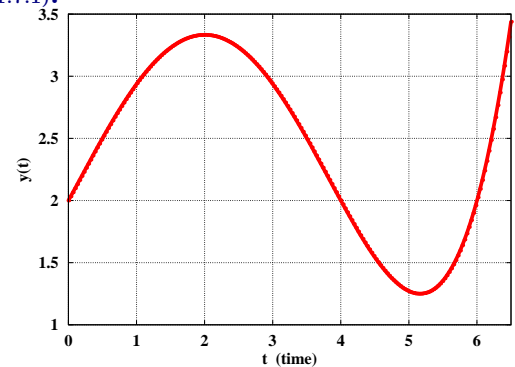
Result: $\frac{d^3y}{dt^3} \triangleq$

5.3 ♣ Geometric interpretation of a derivative. (Section 1.7.1).

Estimate the 1st-derivative of the function $y(t)$ shown to the right at $t = 0, 2, 4, 6$.

Pick your answers from: **-1, 0, 1, 2**.

Result: $\left. \frac{dy}{dt} \right|_{t=0} =$ $\left. \frac{dy}{dt} \right|_{t=2} =$
 $\left. \frac{dy}{dt} \right|_{t=4} =$ $\left. \frac{dy}{dt} \right|_{t=6} =$



Estimate the **sign** of the 2nd-derivative of $y(t)$ from the answers **-**, **0**, or **+**.

Answer **0** when the absolute value of the 2nd-derivative is estimated to be less than 0.5.

Result: $\left. \frac{d^2y}{dt^2} \right|_{t=0}$ is $\left. \frac{d^2y}{dt^2} \right|_{t=2}$ is $\left. \frac{d^2y}{dt^2} \right|_{t=4}$ is $\left. \frac{d^2y}{dt^2} \right|_{t=6}$ is

5.4 ♣ Derivatives of commonly-encountered functions. (Section 1.7.5).

Differentiate the following functions that depend on t (time). Ensure answers involving x are valid when x is either constant or depends on time, e.g., when $x = t^3$.

Result: $\frac{d}{dt} t^2 =$ $\frac{d}{dt} t^3 =$ $\frac{d}{dt} t^{47} =$
 $\frac{d}{dt} \sin(t) =$ $\frac{d}{dt} \cos(t) =$ $\frac{d}{dt} \cos(x) =$
 $\frac{d}{dt} e^t =$ $\frac{d}{dt} \ln(t) =$ $\frac{d}{dt} \ln(x) =$

5.5 ♣ Good product rule for differentiation (for scalars, vectors, matrices, ...). (Section 1.7.7).

The *good product rule for differentiation* that works when u and v are scalars, vectors, or matrices is (circle the correct answer – and update your Calculus teacher):

$$\frac{d(u * v)}{dt} = \frac{du}{dt} * v + u * \frac{dv}{dt} \qquad \frac{d(u * v)}{dt} = u * \frac{dv}{dt} + v * \frac{du}{dt} \qquad \frac{d(u * v)}{dt} = v * \frac{du}{dt} + u * \frac{dv}{dt}$$

Knowing u, v, w are scalars or **matrices** that depend on time t , use the *good product rule for differentiation* to form the 1st ordinary time-derivative of $y(t) = u * v * w$.

Good product rule: $\frac{dy}{dt} = \frac{d(u * v * w)}{dt} =$

5.6 Derivative quotient rule? No, just use product rule and exponent. (Section 1.7.8).

Although the “*quotient rule*” can be used to calculate the derivative with respect to t of the ratio of two functions $\frac{f(t)}{g(t)}$, it can be easier to rewrite the ratio as $f(t) * g(t)^{-1}$ then use the *product rule*. Use this idea to first rewrite the following ratio of two functions as a product and then use the *product rule* to calculate its derivative.

Result: $\frac{\ln(t)}{t^2} =$ $\frac{d}{dt} [\ln(t) / t^2] =$

5.7 ♣ Example of the “good product rule” for differentiation. (Takes less than 2 minutes).

The “good” product rule is easy-to-use for *very quickly* differentiating complex expressions. Knowing x and y are variables that depend on the independent variable t (time), determine the ordinary time-derivative of the function f when¹

$$f(t) = \sin(t) * \cos(x + y) * (\dot{x})^2 * e^t * \ln(y) / x$$

Result:

$$\begin{aligned} \frac{df}{dt} &= \cos(t) * \cos(x + y) * (\dot{x})^2 * e^t * \ln(y) / x \\ &- \sin(t) * \sin(x + y) * (\dot{x} + \dot{y}) * (\dot{x})^2 * e^t * \ln(y) / x \\ &+ \text{} \\ &+ \text{} \\ &+ \text{} \\ &- \text{} \end{aligned}$$

5.8 Differentiation concepts. (Section 1.7.10).

Shown right is an equation relating the dependent variable y to the independent variable t .

$$y^4 - 8y = 3t^2 + \sin(t)$$

Find a general expression for the ordinary derivative $\frac{dy}{dt}$ in terms of t and y .

Find a **numerical** value for $\frac{dy}{dt}$ at $t = 0$ when y is **positive**.

Hint: The value of y is not arbitrary. If you encounter difficulty, consider *implicit differentiation*..

Result:

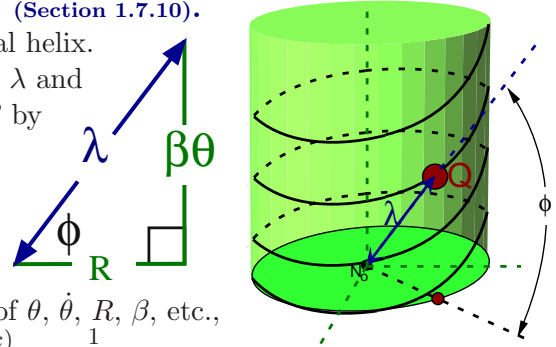
$$\frac{dy}{dt} =$$
 $\left. \frac{dy}{dt} \right|_{t=0} =$

¹Symbols for the 1st and 2nd ordinary time-derivatives of x include $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ (introduced by *Leibniz*), \dot{x} and \ddot{x} (introduced by *Newton*), and \mathbf{x}' and \mathbf{x}'' (introduced by *Lagrange* and used by MotionGenesis).

5.9 Review of explicit and implicit differentiation. (Section 1.7.10).

The figure to the right shows a point Q on a cylindrical helix. Two geometrically significant quantities are a distance λ and an angle ϕ that are related to two constants R and β by

$$\lambda^2 = R^2 + (\beta\theta)^2 \quad \tan(\phi) = \frac{\beta\theta}{R}$$



Determine $\dot{\lambda}$ and $\dot{\phi}$ (time-derivatives of λ and ϕ) in terms of $\theta, \dot{\theta}, R, \beta$, etc., using the two methods described below. Note: $\frac{\partial \text{atan}(x)}{\partial x} = \frac{1}{1+x^2}$

(a) **Explicit differentiation**

Solve explicitly for λ and ϕ and then differentiate the resulting expression.

Result:

$$\lambda = \sqrt{R^2 + (\beta\theta)^2} \quad \phi = \text{atan}\left(\frac{\beta\theta}{R}\right)$$

$$\dot{\lambda} = \text{[Yellow Box]} \quad \dot{\phi} = \text{[Yellow Box]}$$

(b) **Implicit differentiation**

Differentiate the equations involving λ^2 and $\tan(\phi)$ and then solve for $\dot{\lambda}$ and $\dot{\phi}$.

Result:

$$\dot{\lambda} = \text{[Yellow Box]} \quad \dot{\phi} = \text{[Yellow Box]} = \frac{\beta R}{\lambda^2} \dot{\theta}$$

(c) **Explicit/Implicit** differentiation of λ is easier and computationally more efficient.

5.10 ♣ Review of partial and ordinary differentiation. (Section 1.7.2).

The kinetic energy K of a bridge-crane (shown right) can be written in terms of constants M, m, L and variables $x, \dot{x}, \theta, \dot{\theta}$, as

$$K = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m [L^2 \dot{\theta}^2 + 2L \cos(\theta) \dot{x} \dot{\theta}]$$

- First, regard $x, \dot{x}, \theta, \dot{\theta}$ as independent variables [so K depends separately on each, i.e., $K(x, \dot{x}, \theta, \dot{\theta})$], form the **partial derivatives** below (left).
- Next, regard $x, \dot{x}, \theta, \dot{\theta}$ as time-dependent variables and form the **ordinary derivatives** below (right).



The mathematical process below is used in *Lagrange's equations of motion*.

$$\frac{\partial K}{\partial \theta} = \text{[Yellow Box]} \quad \frac{\partial K}{\partial \dot{\theta}} = \text{[Yellow Box]} \quad \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{\theta}} \right) = \text{[Yellow Box]}$$

$$\frac{\partial K}{\partial x} = \text{[Yellow Box]} \quad \frac{\partial K}{\partial \dot{x}} = \text{[Yellow Box]} \quad \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{x}} \right) = \text{[Yellow Box]}$$

5.11 Differentiation concepts: What is dt ? (Section 1.7.1).

A continuous function $z(t)$ depends on $x(t), y(t)$, and time t as

$$z = x + y^2 \sin(t)$$

At a certain instant of time, $y = 1$ and z simplifies to

$$z = x + \sin(t)$$

Find the time-derivative of z at the instant when $y = 1$.

Result:

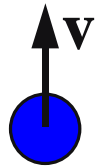
$$\left. \frac{dz}{dt} \right|_{y=1} = \text{[Yellow Box]}$$

5.12 ♣ **Differentiation concepts – what is wrong?** (Section 1.7.1 and previous problem).

The scalar v measures a baseball's upward-velocity. Knowing $v = 0$ only when the ball reaches maximum height, explain what is wrong with the following statement about v 's time derivative.

$$\frac{dv}{dt} = \frac{d(0)}{dt} = 0 \text{ is } \underline{\text{wrong}}. \quad \text{You know the correct answer is: } \frac{dv}{dt} = g \approx 9.8 \frac{\text{m}}{\text{s}^2}.$$

Explain what is wrong: It is incorrect to time-differentiate as shown above because:



5.13 ♣ **Integrals of commonly-encountered functions.** (Section 1.8).

Calculate the following indefinite integrals in terms of an indefinite constant C (regard t as positive).

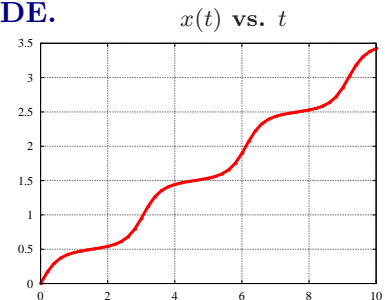
Result:

$\int t^2 dt =$ 	$\int t^3 dt =$ 	$\int t^8 dt =$
$\int t^{-3} dt =$ 	$\int t^{-2} dt =$ 	$\int t^{-1} dt =$
$\int \sin(t) dt =$ 	$\int \cos(t) dt =$ 	$\int e^t dt =$
$\int 5 dt =$ 	$\int 5/t dt =$ 	$\int (5 + \frac{1}{t}) dt =$

5.14 Optional: † **Continuous numerical solution of a nonlinear ODE.**

Plot the continuous solution $x(t)$ to the following ordinary differential equation for $0 \leq t \leq 10$ with data every 0.2 sec. Use an initial value $x(0) = 0$ and use the initial value of \dot{x} that is closest to 1.

$$\sin(\dot{x}) + 4\dot{x}^2 - 1.9 \cos(2\pi x) - 2 = 0$$



Hint: A “clever” way to solve this **nonlinear** ODE for $x(t)$ is

- Use the given equation and initial value $x(0) = 0$ to solve for \dot{x} at $t=0$. For example, the technique in Section 1.11 finds $\dot{x}(t=0) \approx 0.8841161$ when $x(t=0) = 0$.
- Time-differentiate the 1st-order ODE that is **nonlinear** in \dot{x} to form a 2nd-order ODE that is **linear** in \ddot{x} . Then, solve the 2nd-order ODE for \ddot{x} .

$$\cos(\dot{x}) \ddot{x} + 8\dot{x} \ddot{x} + 3.8\pi \sin(2\pi x) \dot{x} = 0 \quad \Rightarrow \quad \ddot{x} = \frac{-3.8\pi \sin(2\pi x) \dot{x}}{\cos(\dot{x}) + 8\dot{x}}$$

- Numerically integrate the 2nd-order ODE with the initial values of $x(0)$ and $\dot{x}(0)$