

Show work – except for ♣ fill-in-blanks.

Angular velocity and angular acceleration

6.1 ♣ FE/EIT Review – Motion graph:

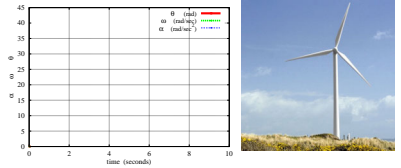
$$\mathbf{T} = \mathbf{I} \alpha \quad \alpha = \frac{d\omega}{dt} \quad \omega = \frac{d\theta}{dt}$$

A wind turbine generates electricity from a torque (created by wind forces) modeled as $T = 20 \frac{\text{Nm}}{\text{sec}} * t$. Equations relating the wind turbine’s angular acceleration α , angular velocity ω , and angle θ are shown above, where moment of inertia $I = 80 \text{ kg m}^2$. Initially, (at $t = 0$), $\omega = 0$ and $\theta = 0$.

Determine $\alpha(t)$, $\omega(t)$, $\theta(t)$.

Result: $\alpha(t) = \frac{\text{[]}}{\text{[]}} \quad \omega(t) = \frac{\text{[]}}{\text{[]}} \quad \theta(t) = \frac{\text{[]}}{\text{[]}}$

Optional: Graph $\alpha(t)$, $\omega(t)$, and $\theta(t)$ for $0 \leq t \leq 10$ sec.



6.2 ♣ Draw a rigid body, center of mass, vector basis, and rigid frame. (Section 7.2.3)

Draw a *rigid body* B , shaped like a uniform-density doughnut (with a hole).

Draw a properly located *center of mass* symbol \ominus and label this point B_{cm} .

Draw a right-handed orthogonal *basis* fixed in B with unit vectors $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3$.

Draw a *rigid frame* with *origin point* B_o (at a location different than B_{cm}) and a right-handed orthogonal *basis* fixed in B with unit vectors $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$.



6.3 ♣ Words and pictures for ${}^b R^a$, ${}^N \vec{\omega}^B$, ${}^N \vec{\alpha}^B$. (Chapters 5 and 7)

Complete each blank below with a phrase to the right.

3D orthonormal basis	rigid body	reference frame
rotation matrix	angular velocity	angular acceleration

${}^a R^b$ R denotes [] a is a [] b is a []	${}^N \vec{\omega}^B$ $\vec{\omega}$ denotes [] ${}^N \vec{\alpha}^B$ $\vec{\alpha}$ denotes [] N is a [] or [] or [] B is a [] or [] or []
<p>Draw a and b.</p>	<p>Draw N and B.</p>

6.4 ♣ Definition of angular velocity? (Section 7.3.3).

A “popular” definition of angular velocity is $\vec{\omega} \triangleq \dot{\theta} \vec{\mathbf{k}}$. This definition is generally useful for calculating angular velocity and proving its properties (for both 2D and/or 3D analysis). **True/False**

6.5 ♣ Optional: Textbook/Internet definition of 3D angular velocity (Section 7.3).

Renowned dynamicist Thomas Kane said angular velocity is one of the “**most misunderstood concepts in kinematics.**” Report a definition of angular velocity and determine if the quantities appearing in the definition are **rigorously defined** - and whether the definition is generally applicable for 3D kinematics or only applies for simple angular velocity (e.g., $\vec{\omega} = \dot{\theta} \vec{\mathbf{k}}$ described in Section 7.3.3).

Note: A definition should be able to **prove** important theorems [such as the angular velocity addition theorem of equation (7.4) and the golden rule for vector differentiation in equation (7.1)] and allow for angular velocity **calculations**.

Source (reference). List textbook or .html link	Definition. Report the defining equation/property.	Rigorously defined?	Works for 3D kinematics?
[]	[]	Yes/No	Yes/No

6.6 ♣ **Angular velocity of a Ferris-wheel seat** (courtesy of David Levinson).

Known: The rigid seat on a Ferris wheel does not change its orientation relative to ground as the Ferris wheel rotates.

Decide: The seat's angular velocity $\vec{\omega}$ relative to ground (circle one):

Is zero $\vec{\omega} = \vec{0}$	Is constant $\vec{\omega} = \text{Constant} \neq \vec{0}$	Varies $\vec{\omega} = \vec{\omega}(t)$	Does not exist
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6.7 ♣ **Concept: What objects have a unique angular velocity/acceleration?** (Sections 7.3, 7.4).

The angular velocity $\vec{\omega}$ of some object S relative to Earth is to be determined.

This object S could be a (circle **all** objects that have an **unambiguously** defined angular velocity $\vec{\omega}$):

Real number	Matrix	Set of points	Mass center of a rigid body
Vector	Point	Reference frame	Flexible body
3D orthogonal unit basis	Particle	3D rigid body	System of particles and bodies

Repeat for the angular acceleration $\vec{\alpha}$ of some object S relative to Earth appropriate objects.

6.8 ♣ **Concepts: What objects have a uniquely-defined angular velocity?** (Section 7.3).

a. The angular velocity of a point	in a reference frame	is well defined.	True/False
b. The angular velocity of a 3D rigid body	in a particle	is well defined.	True/False
c. The angular velocity of a 3D rigid body	in a reference frame	is well defined.	True/False
d. The angular velocity of a reference frame	in a 3D rigid body	is well defined.	True/False
d. The angular velocity of a 3D rigid basis	in a reference frame	is well defined.	True/False
e. The angular velocity of a reference frame	in a flexible body	is well defined.	True/False
f. The angular velocity of a flexible body	in a reference frame	is well defined.	True/False

6.9 ♣ **Memorize the “golden rule for vector differentiation”.** (Section 7.3).

The angular velocity ${}^A\vec{\omega}^B$ characterizes the time-rate of change of orientation of a reference frame (or rigid vector basis) B in a reference frame (or rigid vector basis) A .

★ **Very important formula** ★

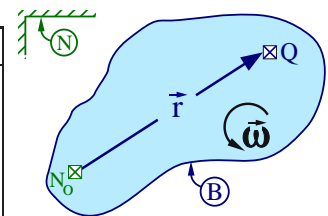
Complete the defining property of ${}^A\vec{\omega}^B$ shown right which relates time-derivatives of **any vector** \vec{v} in A and B .

$$\frac{{}^A d\vec{v}}{dt} \stackrel{(7.1)}{=} \frac{{}^B d\vec{v}}{dt} + \text{[]} \times \text{[]}$$

6.10 ♣ **Vector differentiation concepts** “ $v = \omega r$ ”. (Section 7.3).

Point Q is fixed on a rigid body B . Point N_o is fixed in a reference frame N and does not move on B . Complete the following proof that shows how \vec{v} (Q 's velocity in N) can be written in terms of ${}^N\vec{\omega}^B$ (B 's angular velocity in N) and \vec{r} (Q 's position from N_o).

Mathematical statement	Reasoning (explain each step below with a brief phrase)
$\vec{v} \triangleq \frac{{}^N d\vec{r}}{dt}$	Definition of Q 's velocity in N
$= \text{[]} + \text{[]}$	<input type="text"/>
$= \vec{0} + {}^N\vec{\omega}^B \times \vec{r}$	<input type="text"/>



6.11 ♣ **Rotational kinematics of a fire ladder.** (Sections 7.3.3, 7.3.5, 7.3.6).

The following figure shows a fire truck chassis A traveling at constant speed in straight-line motion on Earth (A does not rotate relative to Earth). Earth is a **Newtonian reference frame** N .

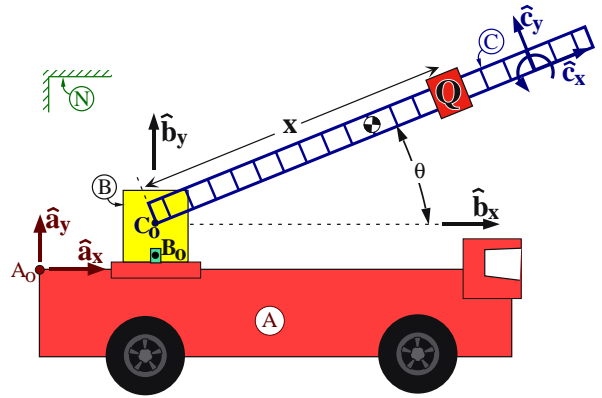
A rigid hub B is connected to fire truck A by a revolute motor at point B_o of B .

A rigid ladder C is connected to hub B by a revolute motor at point C_o of C .

A fire-fighter Q (modeled as a particle of mass m) climbs ladder C .

Right-handed orthogonal unit vectors $\hat{a}_x, \hat{a}_y, \hat{a}_z$; $\hat{b}_x, \hat{b}_y, \hat{b}_z$; $\hat{c}_x, \hat{c}_y, \hat{c}_z$; are fixed in A, B, C , with:

- \hat{a}_x pointing forward on the fire truck
- \hat{a}_y vertically-upward and from B_o to C_o
- $\hat{b}_y = \hat{a}_y$ parallel to the axis of the revolute motor that connects B and A
- $\hat{b}_z = \hat{c}_z$ parallel to the axis of the revolute motor that connects B and C
- \hat{c}_x directed from C_o to Q (along the ladder)



Note: **Visualize** C 's "Body yz " (or "Space zy ") rotation sequence in N (e.g., with a ruler).

Quantity	Symbol	Type
\hat{b}_y measure of B 's angular velocity in A	ω_B	Constant
Angle from \hat{b}_x to \hat{c}_x with $+\hat{b}_z$ sense	θ	Variable

${}^cR^b$			

- (a) Complete the previous ${}^cR^b$ rotation table (to the right).
Note: ${}^cR^b$ is unnecessary for the remainder of this problem.
- (b) Clarify the process to determine C 's angular velocity in B and ${}^A\vec{\omega}^B$. (Section 7.3.3).
- ${}^B\vec{\omega}^C$ is a **simple** angular velocity because \hat{b}_z is a vector fixed in **both** and .
 - ${}^B\vec{\omega}^C = \text{[]}$. The sign (\pm) is determined using the -hand rule (sweep from to .
 - ${}^A\vec{\omega}^B = \omega_B \hat{b}_y$ is a **simple** angular velocity because \hat{b}_y is fixed in **both** and .

- (c) Form C 's angular velocity in N and express it in terms of $\hat{b}_x, \hat{b}_y, \hat{b}_z$. (Section 7.3.5).

Result:

$${}^N\vec{\omega}^C \stackrel{(7.4)}{=} \text{[]}\vec{\omega}^{\text{[]}} + \text{[]}\vec{\omega}^{\text{[]}} + \text{[]}\vec{\omega}^{\text{[]}} = \vec{0} + \text{[]}\hat{b}_y + \text{[]}\hat{b}_z$$

- (d) When both ω_B and $\dot{\theta}$ are **constant**, ${}^N\vec{\alpha}^C = \vec{0}$. **True/False**.

- (e) Write the definition for C 's angular acceleration in N and form ${}^N\vec{\alpha}^C$. (Sections 7.4, 7.3).

Result:

$${}^N\vec{\alpha}^C \stackrel{(7.5)}{\triangleq} \text{[]} \quad {}^N\vec{\alpha}^C \stackrel{(7.1)}{=} \omega_B \dot{\theta} \hat{b}_x + \ddot{\theta} \hat{b}_z$$

6.12 ♣ Theorems: Rotation matrices R , angular velocity $\vec{\omega}$, angular acceleration $\vec{\alpha}$? (Section 7.4).

Determine the validity of each theorem to the right for general 3D motion of rigid frames A, B, C, D .

Note: Rigid basis a contains right-handed orthogonal unit vectors $\hat{a}_x, \hat{a}_y, \hat{a}_z$ fixed in rigid frame A .

Similarly for rigid bases b, c, d fixed in frames B, C, D .

Theorem	True or false?
${}^aR^d = {}^aR^b * {}^bR^c * {}^cR^d$	True/False
${}^A\vec{\omega}^D = {}^A\vec{\omega}^B + {}^B\vec{\omega}^C + {}^C\vec{\omega}^D$	True/False
${}^A\vec{\alpha}^D = {}^A\vec{\alpha}^B + {}^B\vec{\alpha}^C + {}^C\vec{\alpha}^D$	True/False

6.13 Alternate formula for angular acceleration. (Section 7.3).

Show/prove ${}^N\vec{\alpha}^B \triangleq \frac{N_d N\vec{\omega}^B}{dt}$ can also be calculated as ${}^N\vec{\alpha}^B = \frac{B_d N\vec{\omega}^B}{dt}$.

6.14 Optional: Angular acceleration addition theorem. (Sections 7.3, 7.3.5, 7.4).

Use the angular velocity addition theorem and the definition of angular acceleration to prove:

$${}^N\vec{\alpha}^B = {}^N\vec{\alpha}^A + {}^A\vec{\alpha}^B + N\vec{\omega}^A \times A\vec{\omega}^B$$