5.1 ♣ Notations for derivatives. (Section 1.6.1).

Date	Person	Symbols for 1^{st}	2^{nd}, and	3^{rd} derivatives
1675		$\frac{dy}{dt}$	$\frac{d^2y}{dt^2}$	$\frac{d^3y}{dt^3}$
1675		$\dot{y}$	$\ddot{y}$	$\dddot{y}$
1797	(trained by Euler)	y'	y''	y'''
1850	Cauchy/Weierstrauss	$\lim\limits_{h \to 0} \frac{y(t+h) - y(t)}{h}$	?	?
1786	Legendre (introduced partials then abandoned)	$\frac{\partial y}{\partial x}$	$\frac{\partial^2 y}{\partial x^2}$	$\frac{\partial^3 y}{\partial x^3}$
1841	Jacobi (re-introduced partials again)			

There was bitter rivalry between Newton and Leibniz, and the notations of Leibniz and Newton are not entangled.
For example, $\frac{d\dot{y}}{dt}$ is written in Leibniz's notation as   or Newton's as   .

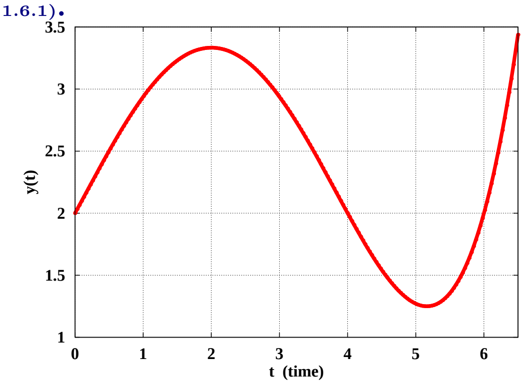
5.2 ♣ Leibniz's shorthand notation for 3^{rd} derivatives. (Section 1.6.1).

Write the explicit expression for the following 3^{rd} derivative (so it contains three 1^{st} derivatives).

Result:

$$\frac{d^3y}{dt^3} \triangleq$$

5.3 ♣ Geometric interpretation of a derivative. (Section 1.6.1).



Estimate the 1^{st}-derivative of the function $y(t)$ shown to the right at $t = 0,\ 2,\ 4,\ 6$.

Pick your answers from: **-1**, **0**, **1**, **2**.

Result: $\left.\frac{dy}{dt}\right|_{t=0} =$ $\left.\frac{dy}{dt}\right|_{t=2} =$

$\left.\frac{dy}{dt}\right|_{t=4} =$ $\left.\frac{dy}{dt}\right|_{t=6} =$

Estimate the <u>sign</u> of the 2^{nd}-derivative of $y(t)$ from the answers **−**, **0**, or **+**.
Answer **0** when the absolute value of the 2^{nd}-derivative is estimated to be less than 0.5.

Result: $\left.\frac{d^2y}{dt^2}\right|_{t=0}$ is $\left.\frac{d^2y}{dt^2}\right|_{t=2}$ is $\left.\frac{d^2y}{dt^2}\right|_{t=4}$ is $\left.\frac{d^2y}{dt^2}\right|_{t=6}$ is

5.4 ♣ Derivatives of commonly-encountered functions. (Section 1.6.5).

Differentiate the following functions that depend on t (time). Ensure answers involving x are valid when x is either constant or depends on time, e.g., when $x = t^3$.

Result: $\frac{d}{dt}\, t^2 =$ $\frac{d}{dt}\, t^3 =$ $\frac{d}{dt}\, t^{47} =$

$\frac{d}{dt}\, \sin(t) =$ $\frac{d}{dt}\, \cos(t) =$ $\frac{d}{dt}\, \cos(x) =$

$\frac{d}{dt}\, e^t =$ $\frac{d}{dt}\, \ln(t) =$ $\frac{d}{dt}\, \ln(x) =$

5.5 ♣ Good product rule for differentiation (for scalars, $\vec{\text{v}}$ectors, matrices, ...). (Section 1.6.7).

The ***good product rule for differentiation*** that works when u and v are scalars, $\vec{\text{v}}$ectors, or matrices is (circle the correct answer – and update your Calculus teacher):

$$\frac{d(u*v)}{dt} = \frac{du}{dt} * v + u * \frac{dv}{dt} \qquad \frac{d(u*v)}{dt} = u * \frac{dv}{dt} + v * \frac{du}{dt} \qquad \frac{d(u*v)}{dt} = v * \frac{du}{dt} + u * \frac{dv}{dt}$$

Knowing u, v, w are scalars or **matrices** that depend on time t, use the ***good product rule for differentiation*** to form the 1^{st} ordinary time-derivative of $y(t) = u * v * w$.

Good product rule: $\frac{dy}{dt} = \frac{d(u*v*w)}{dt} =$

5.6 Derivative quotient rule? No, just use product rule and exponent. (Section 1.6.8).

Although the "***quotient rule***" can be used to calculate the derivative with respect to t of the ratio of two functions $\frac{f(t)}{g(t)}$, it can be easier to rewrite the ratio as $f(t) * g(t)^{-1}$ then use the ***product rule***. Use this idea to first rewrite the following ratio of two functions as a product and then use the ***product rule*** to calculate its derivative.

Result: $\frac{\ln(t)}{t^2} =$ $\qquad$ $\frac{d}{dt}\left[\ln(t) / t^2\right] =$

5.7 ♣ Example of the "good product rule" for differentiation. (Takes less than 2 minutes).

The "good" product rule is easy-to-use for ***very quickly*** differentiating complex expressions. Knowing x and y are variables that depend on the independent variable t (time), determine the ordinary time-derivative of the function f when[1]

$$f(t) = \sin(t) * \cos(x+y) * (\dot{x})^2 * e^t * \ln(y) / x$$

Result:

$$\begin{aligned} \frac{df}{dt} &= \cos(t) * \cos(x+y) * (\dot{x})^2 * e^t * \ln(y) / x \\ &- \sin(t) * \sin(x+y) * (\dot{x}+\dot{y}) * (\dot{x})^2 * e^t * \ln(y) / x \\ &+ \\ &+ \\ &+ \\ &- \end{aligned}$$

5.8 ♣ Ordinary derivative of the function $f(t) = \sin(t) * \cos(x\,y\,z)$. (Sections 1.6.7 and 1.6.9).

Knowing each of x, y, z depend on time t, form the 1^{st}-derivative of $f(t)$ (in terms of x, y, z, t, etc).

Result: $\frac{d\,[\sin(t)\,\cos(x\,y\,z)]}{dt} =$

[1]Symbols for the 1^{st} and 2^{nd} ordinary time-derivatives of x include $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ (introduced by ***Leibniz***), $\dot{x}$ and $\ddot{x}$ (introduced by ***Newton***), and `x'` and `x''` (introduced by ***Lagrange*** and used by **M**otion**G**enesis).

5.9 Differentiation concepts. (Section 1.6.10).

Shown right is an equation relating the dependent variable y to the independent variable t.

$$y^4 \;-\; 8\,y \;=\; 3\,t^2 \;+\; \sin(t)$$

Find a general expression for the ordinary derivative $\frac{dy}{dt}$ in terms of t and y.

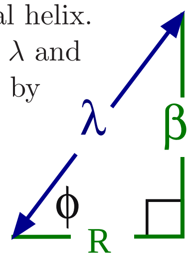
Find a **numerical** value for $\frac{dy}{dt}$ at $t = 0$ when y is **positive**.

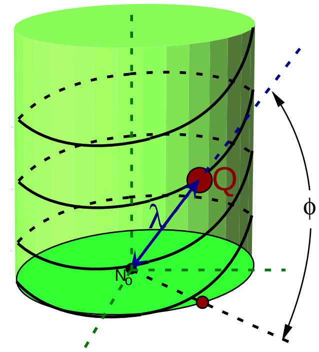
Hint: The value of y is not arbitrary. If you encounter difficulty, consider ***implicit differentiation***..

Result: $\quad \dfrac{dy}{dt} = \boxed{} \qquad \left.\dfrac{dy}{dt}\right|_{t=0} = \boxed{}$

5.10 Review of explicit and implicit differentiation. (Section 1.6.10).

The figure to the right shows a point Q on a cylindrical helix. Two geometrically significant quantities are a distance λ and an angle ϕ that are related to two **constants** R and β by





$$\lambda^2 \;=\; R^2 \;+\; (\beta\,\theta)^2 \qquad\qquad \tan(\phi) \;=\; \frac{\beta\,\theta}{R}$$

Determine $\dot{\lambda}$ and $\dot{\phi}$ (time-derivatives of λ and ϕ) in terms of θ, $\dot{\theta}$, R, β, etc., using the two methods described below. Note: $\dfrac{\partial\,\text{atan}(x)}{\partial x} \;=\; \dfrac{1}{1+x^2}$

(a) ***Explicit differentiation***

Solve explicitly for λ and ϕ and then differentiate the resulting expression.

Result: $\quad \lambda \;=\; \sqrt{R^2 \;+\; (\beta\,\theta)^2} \qquad \phi \;=\; \text{atan}(\,\frac{\beta\,\theta}{R}\,)$

$\dot{\lambda} \;=\; \boxed{} \qquad \dot{\phi} \;=\; \boxed{}$

(b) ***Implicit differentiation***

Differentiate the equations involving λ^2 and $\tan(\phi)$ and then solve for $\dot{\lambda}$ and $\dot{\phi}$.

Result: $\quad \dot{\lambda} \;=\; \boxed{} \qquad \dot{\phi} \;=\; \boxed{} \;=\; \frac{\beta\,R}{\lambda^2}\,\dot{\theta}$

(c) **Explicit/Implicit** differentiation of λ is easier and computationally more efficient.

5.11 ♣ Review of partial and ordinary differentiation. (Section 1.6.2).

The kinetic energy K of a bridge-crane (shown right) can be written in terms of constants M, m, L and variables x, $\dot{x}$, θ, $\dot{\theta}$, as

$$K \;=\; \tfrac{1}{2}\,M\,\dot{x}^2 \;\;+\;\; \tfrac{1}{2}\,m\,[L^2\,\dot{\theta}^2 + 2\,L\,\cos(\theta)\,\dot{x}\,\dot{\theta}]$$

The mathematical process below is used in ***Lagrange's equations of motion***.

- First, regard x, $\dot{x}$, θ, $\dot{\theta}$ as independent variables [so K depends separately on each, i.e., $K(x,\,\dot{x},\,\theta,\,\dot{\theta})$], form the **partial derivatives** below (left).
- Next, regard x, $\dot{x}$, θ, $\dot{\theta}$ as time-dependent variables and form the **ordinary derivatives** below (right).

$$\frac{\partial K}{\partial \theta} = \boxed{} \qquad \frac{\partial K}{\partial \dot{\theta}} = \boxed{} \qquad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}}\right) = \boxed{}$$

$$\frac{\partial K}{\partial x} = \boxed{} \qquad \frac{\partial K}{\partial \dot{x}} = \boxed{} \qquad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) = \boxed{}$$

5.12 Differentiation concepts: What is dt? (Section 1.6.1).

A continuous function $z(t)$ depends on $x(t)$, $y(t)$, and time t as $\quad z \;=\; x \;+\; y^2 \sin(t)$

At a certain instant of time, $y = 1$ and z simplifies to $\quad z \;=\; x \;+\; \sin(t)$

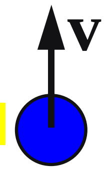
Find the time-derivative of z at the instant when $y = 1$.

Result: $\quad \left.\dfrac{dz}{dt}\right|_{y=1} \;=$

5.13 ♣ Differentiation concepts – what is wrong? (Section 1.6.1 and previous problem).

The scalar v measures a baseball's upward-velocity. Knowing $v = 0$ only when the ball reaches maximum height, explain what is wrong with the following statement about v's time derivative.

$\dfrac{dv}{dt} \;=\; \dfrac{d\,(0)}{dt} \;=\; 0$ is <u>wrong</u>. You know the correct answer is: $\dfrac{dv}{dt} \;=\; g \approx 9.8\,\dfrac{\text{m}}{\text{s}^2}$.



Explain what is <u>wrong</u>: It is incorrect to time-differentiate as shown above because:

5.14 ♣ Integrals of commonly-encountered functions. (Section 1.7).

Calculate the following indefinite integrals in terms of an indefinite constant C (regard t as positive).

Result:

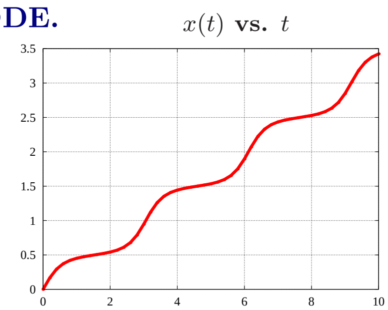
$\int t^2\, dt \;=$ $\qquad \int t^3\, dt \;=$ $\qquad \int t^8\, dt \;=$

$\int t^{-3}\, dt \;=$ $\qquad \int t^{-2}\, dt \;=$ $\qquad \int t^{-1}\, dt \;=$

$\int \sin(t)\, dt \;=$ $\qquad \int \cos(t)\, dt \;=$ $\qquad \int e^t\, dt \;=$

$\int 5\, dt \;=$ $\qquad \int 5/t\, dt \;=$ $\qquad \int (5 + \frac{1}{t})\, dt \;=$

5.15 Optional: † Continuous numerical solution of a nonlinear ODE.



Plot the continuous solution $x(t)$ to the following ordinary differential equation for $0 \le t \le 10$ with data every 0.2 sec. Use an initial value $x(0) = 0$ and use the initial value of $\dot{x}$ that is closest to 1.

$$\sin(\dot{x}) \;+\; 4\,\dot{x}^2 \;-\; 1.9\,\cos(2\,\pi\,x) \;-\; 2 \;=\; 0$$

Hint: A "clever" way to solve this **nonlinear** ODE for $x(t)$ is

- Use the given equation and initial value $x(0) = 0$ to solve for $\dot{x}$ at $t\!=\!0$.
  For example, the technique in Section 1.10 finds $\dot{x}(t\!=\!0) \approx 0.8841161$ when $x(t\!=\!0) = 0$.
- Time-differentiate the 1^{st}-order ODE that is **nonlinear** in $\dot{x}$ to form a 2^{nd}-order ODE that is **linear** in $\ddot{x}$.
  Then, solve the 2^{nd}-order ODE for $\ddot{x}$.

$$\cos(\dot{x})\,\ddot{x} \;+\; 8\,\dot{x}\,\ddot{x} \;+\; 3.8\,\pi\,\sin(2\,\pi\,x)\,\dot{x} \;=\; 0 \qquad \Rightarrow \qquad \ddot{x} \;=\; \frac{-3.8\,\pi\,\sin(2\,\pi\,x)\,\dot{x}}{\cos(\dot{x}) \;+\; 8\,\dot{x}}$$

- Numerically integrate the 2^{nd}-order ODE with the initial values of $x(0)$ and $\dot{x}(0)$
