5.1 ♣ Notations for derivatives (complete the blanks). (Section 1.6.1).

Date	Name of mathematician	Idea	Symbols for 1^{st},	2^{nd},	3^{rd} derivatives
1675	$\boxed{}$	Differentials	$\frac{dy}{dt}$	$\boxed{}$	$\boxed{}$
1675	$\boxed{}$	Geometry/slope	$\dot{y}$	$\boxed{}$	$\boxed{}$
1797	$\boxed{}$ (trained by Euler)	Functions	y'	$\boxed{}$	$\boxed{}$
1850	Cauchy (trained by Lagrange)	Limits	$\lim\limits_{h \to 0} \frac{y(t+h) - y(t)}{h}$	?	?
1786	Legendre (introduced partials then abandoned)		$\frac{\partial y}{\partial x}$	$\frac{\partial^2 y}{\partial x^2}$	$\frac{\partial^3 y}{\partial x^3}$
1841	Jacobi (re-introduced partials again)				

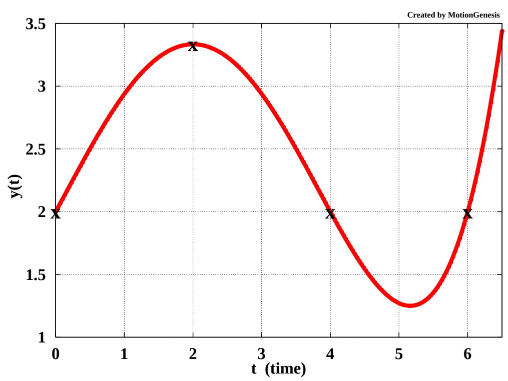
There was bitter rivalry between Newton and Leibniz, and the notations of Leibniz and Newton are not entangled. For example, $\frac{d\dot{y}}{dt}$ is written in Leibniz's notation as $\frac{d^2y}{dt^2}$ or in Newton's notation as $\ddot{y}$.

5.2 ♣ Leibniz's shorthand notation for 3^{rd} derivatives. (Section 1.6.1).

Write the explicit expression for Leibniz's 3^{rd} derivative show right (so it contains three 1^{st} derivatives).

$$\frac{d^3y}{dt^3} \triangleq \frac{d}{dt}\left(\boxed{}\left(\boxed{}\right)\right)$$

5.3 ♣ Newton's idea: Derivative as geometry (slope and curvature). (Section 1.6.1).



Newton related derivatives to geometry (1^{st}-derivative as slope and 2^{nd}-derivative as curvature). Estimate the slope of the function $y(t)$ shown right at $t = 0, 2, 4, 6$.

Pick your answers from: **-1**, **0**, **1**, **2**.

Result: $\left.\frac{dy}{dt}\right|_{t=0} = \boxed{}$ $\left.\frac{dy}{dt}\right|_{t=2} = \boxed{}$

$\left.\frac{dy}{dt}\right|_{t=4} = \boxed{}$ $\left.\frac{dy}{dt}\right|_{t=6} = \boxed{}$

Estimate the sign of the curvature [2^{nd}-derivative of $y(t)$] from the answers **−**, **0**, or **+**. Answer **0** when your estimate of the absolute value of curvature is less than 0.5.

Result: $\left.\frac{d^2y}{dt^2}\right|_{t=0}$ is $\boxed{}$ $\left.\frac{d^2y}{dt^2}\right|_{t=2}$ is $\boxed{}$ $\left.\frac{d^2y}{dt^2}\right|_{t=4}$ is $\boxed{}$ $\left.\frac{d^2y}{dt^2}\right|_{t=6}$ is $\boxed{}$

5.4 ♣ Euler's idea: Derivative of a function is a function. (Section 1.6.5).

Differentiate the following functions that depend on t (time). Express results in terms of x, $\dot{x}$, t so the results are valid when x is constant or depends on time (e.g., when $x = t^3$).

Result:

$\frac{d}{dt}\, t^2 = \boxed{}$ $\frac{d}{dt}\, t^3 = \boxed{}$ $\frac{d}{dt}\, t^{47} = \boxed{}$

$\frac{d}{dt}\, \sin(t) = \boxed{}$ $\frac{d}{dt}\, \cos(t) = \boxed{}$ $\frac{d}{dt}\, \cos(x) = \boxed{}$

$\frac{d}{dt}\, e^t = \boxed{}$ $\frac{d}{dt}\, \ln(t) = \frac{\boxed{}}{\boxed{}}$ $\frac{d}{dt}\, \ln(x) = \frac{1}{\boxed{}} * \boxed{}$

5.5 ♣ Good product rule for differentiation (for scalars, $\vec{\mathbf{v}}$ectors, matrices, ...). (Section 1.6.7).

The ***good product rule for differentiation*** that works when u and v are scalars, $\vec{\mathbf{v}}$ectors, or matrices is (circle the correct answer – and update your Calculus teacher):

$$\frac{d(u*v)}{dt} = \frac{du}{dt}*v + u*\frac{dv}{dt} \qquad \frac{d(u*v)}{dt} = u*\frac{dv}{dt} + v*\frac{du}{dt} \qquad \frac{d(u*v)}{dt} = v*\frac{du}{dt} + u*\frac{dv}{dt}$$

Knowing u, v, w are scalars or **matrices** that depend on time t, use the ***good product rule for differentiation*** to form the 1^{st} ordinary time-derivative of $y(t) = u*v*w$.

Good product rule: $\frac{dy}{dt} = \frac{d(u*v*w)}{dt} = \boxed{} + \boxed{} + \boxed{}$

5.6 ♣ Example of the "good product rule" for differentiation. (Done right, takes less than 2 minutes).

Determine the ordinary derivative of the function $f(t)$ with the easy-to-use ***good product rule for differentiation***. Note: The "good product rule" is an ***efficient*** way to differentiate multiple-factor expressions.

$$f(t) = \sin(t) * \cos(t) * t^2 * e^t * \ln(t)$$

Result:

$$\begin{aligned}\frac{df}{dt} &= \cos(t) * \cos(t) * t^2 * e^t * \ln(t)\\ &- \sin(t) * \boxed{} * t^2 * e^t * \ln(t)\\ &+ \sin(t) * \boxed{} * \boxed{} * \boxed{} * \boxed{}\\ &+ \sin(t) * \boxed{} * \boxed{} * \boxed{} * \boxed{}\\ &+ \sin(t) * \boxed{} * \boxed{} * \boxed{} * \boxed{}\end{aligned}$$

5.7 ♣ Derivative quotient rule? No, just use product rule and exponent. (Section 1.6.8).

For two functions $f(t)$ and $g(t)$, the "***quotient rule***" calculates the derivative of the ratio $\frac{f(t)}{g(t)}$.

However, it can be easier to rewrite this ratio as $f(t) * g(t)^{-1}$ and then use the ***product rule***.

Use this ***product rule*** idea to calculate the derivative below.

Result: $\frac{\sin(t)}{t^2} = \sin(t) * t^{\boxed{}} \qquad \frac{d}{dt}\left[\sin(t) * t^{\boxed{}}\right] = \boxed{}$

5.8 ♣ Chain rule for differentiation. (Section 1.6.9)

Knowing $x(t)$ and $y(t)$ are variables that depend on the independent variable t (time), determine the ordinary time-derivative of the function $f(t)$.

$$f(t) = \sin(x) + \cos(x+y) + (\dot{x})^2 + e^x + \ln(y) + \tfrac{1}{x}$$

Result: $\frac{df}{dt} = \cos(x)\boxed{} - \boxed{} + \boxed{} + \boxed{} + \boxed{} - \boxed{}$

5.9 ♣ Ordinary derivative of the function $f(t) = \sin(t) * \cos(x\,y\,z)$. (Sections 1.6.7 and 1.6.9).

Knowing each of x, y, z depend on time t, form the 1^{st}-derivative of $f(t)$ (in terms of x, y, z, t, etc).

Result: $\frac{d\,[\sin(t)\,\cos(x\,y\,z)]}{dt} = \boxed{}$

5.10 ♣ Differentiation concepts. (Section 1.6.10 – implicit differentiation).

$$y^4 \;-\; 8\,y \;=\; 3\,t^2 \;+\; \sin(t)$$

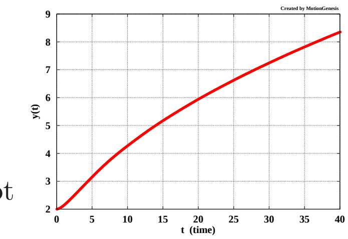
Roots: $y = \boxed{}$, $y = \boxed{}$, $y \approx$ -1 $\pm$ 1.732 i

Shown right is an equation relating the dependent variable $y(t)$ to the independent variable t. Find two real roots to this equation when $t = 0$.

Form a general expression for $\frac{dy}{dt}$ and calculate $\frac{dy}{dt}$ at $t = 0$ (**numerical** value) when y is **positive**.

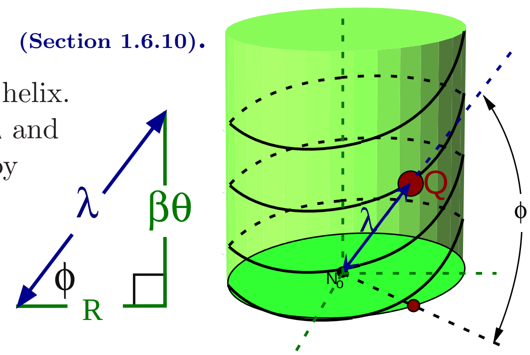
Result:

In terms of t and y: $\quad \dfrac{dy}{dt} = \dfrac{\boxed{}}{\boxed{}} \qquad \left.\dfrac{dy}{dt}\right|_{t=0} = \dfrac{1}{\boxed{}}$



†Optional: Continuous solution of nonlinear algebraic equation.
Starting with $y = 2$, continuously solve for $y(t)$ for $0 \le t \le 40$ and plot your results as shown right. Stumped: See hint in Homework 5.18.

5.11 ♣ Review of explicit and implicit differentiation. (Section 1.6.10).



The figure to the right shows a point Q on a cylindrical helix. Two geometrically significant quantities are a distance λ and an angle ϕ that are related to two **constants** R and β by

$$\lambda^2 \;=\; R^2 \;+\; (\beta\,\theta)^2 \qquad\qquad \tan(\phi) \;=\; \frac{\beta\,\theta}{R}$$

Form $\dot\lambda$ and $\dot\phi$ using the two methods described below.

Hint: $\dfrac{\partial\, \text{atan}(x)}{\partial x} \;=\; \dfrac{1}{1+x^2}$

(a) ***Explicit differentiation***
Solve explicitly for λ and ϕ and then differentiate the resulting expression.
Result:
In terms of R, β, θ, $\dot\theta$.

$$\lambda \;=\; \sqrt{R^2 \;+\; (\beta\,\theta)^2} \qquad\qquad \phi \;=\; \text{atan}(\tfrac{\beta\,\theta}{R})$$

$$\dot\lambda \;=\; \frac{\boxed{}}{\boxed{}}\,\dot\theta \qquad\qquad \dot\phi \;=\; \frac{\boxed{}}{\boxed{}}\,\dot\theta$$

(b) ***Implicit differentiation***
Differentiate the equations involving λ^2 and $\tan(\phi)$ and then solve for $\dot\lambda$ and $\dot\phi$.
Result:
In terms of R, β, θ, $\dot\theta$, λ.

$$\dot\lambda \;=\; \frac{\boxed{}}{\boxed{}}\,\dot\theta \qquad\qquad \dot\phi \;=\; \boxed{}\,\dot\theta \;=\; \frac{\beta\,R}{\lambda^2}\,\dot\theta$$

(c) **Explicit/Implicit** differentiation of λ is easier and computationally more efficient.

5.12 ♣ Review of partial and ordinary differentiation. (Section 1.6.2).

The kinetic energy K of a bridge-crane (shown right) can be written in terms of constants M, m, L and variables x, $\dot x$, θ, $\dot\theta$, as

$$\text{K} \;=\; \tfrac{1}{2}\,M\,\dot x^2 \;+\; \tfrac{1}{2}\,m\,[L^2\,\dot\theta^2 + 2\,L\,\cos(\theta)\,\dot x\,\dot\theta]$$

- First, regard x, $\dot x$, θ, $\dot\theta$ as independent variables [so K depends separately on each, i.e., K(x, $\dot x$, θ, $\dot\theta$)], form the **partial derivatives** below (left).
- Next, regard x, $\dot x$, θ, $\dot\theta$ as time-dependent variables and form the **ordinary derivatives** below (right).

The mathematical process below is used in *Lagrange's equations of motion*.

$\frac{\partial \text{K}}{\partial \theta} = \boxed{}$	$\frac{\partial \text{K}}{\partial \dot\theta} = \boxed{}$	$\frac{d}{dt}\left(\frac{\partial \text{K}}{\partial \dot\theta}\right) = \boxed{}$
$\frac{\partial \text{K}}{\partial x} = \boxed{}$	$\frac{\partial \text{K}}{\partial \dot x} = \boxed{}$	$\frac{d}{dt}\left(\frac{\partial \text{K}}{\partial \dot x}\right) = \boxed{}$

5.13 ♣ **Leibniz's idea and differentiation concepts: What is dt?** (Section 1.6.1).

A continuous function $z(t)$ depends on $x(t)$, $y(t)$, and time t as $\quad z = x + y^2 \sin(t)$

At a certain instant of time, $y = 1$ and z simplifies to $\quad z = x + \sin(t)$

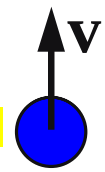
Determine the time-derivative of z at the instant when $y = 1$.

Result: $\quad \left.\frac{dz}{dt}\right|_{y=1} = \boxed{\qquad\qquad}$

5.14 ♣ **Differentiation concepts – what is wrong?** (Section 1.6.1 and previous problem).

The scalar v measures a baseball's upward-velocity. Knowing $v = 0$ only when the ball reaches maximum height, explain what is wrong with the following statement about v's time derivative.

$\frac{dv}{dt} = \frac{d\,(0)}{dt} = 0$ is **wrong** $\quad$ We know the correct answer is: $\frac{dv}{dt} = g \approx 9.8\,\frac{\text{m}}{\text{s}^2}$.



Explain what is wrong: It is incorrect to time-differentiate as shown above because:

$\boxed{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$

5.15 ♣ **Euler's idea: Integral of a function is a function.** (Section 1.7).

Calculate the following indefinite integrals in terms of an indefinite constant C (regard t as positive).

Result:

$\int t^2\,dt = \boxed{\quad} + C \qquad \int t^3\,dt = \boxed{\quad} + C \qquad \int t^8\,dt = \boxed{\quad} + \boxed{\quad}$

$\int t^{-3}\,dt = \boxed{\qquad} \qquad \int t^{-2}\,dt = \boxed{\qquad} \qquad \int t^{-1}\,dt = \boxed{\qquad}$

$\int \sin(t)\,dt = \boxed{\qquad} \qquad \int \cos(t)\,dt = \boxed{\qquad} \qquad \int e^t\,dt = \boxed{\qquad}$

$\int 5\,dt = \boxed{\qquad} \qquad \int 5/t\,dt = \boxed{\qquad} \qquad \int (5 + \frac{1}{t})\,dt = \boxed{\qquad}$

5.16 ♣ **Solve a 3rd-order ODE with mixed initial/boundary values.** (Section 1.7).

Solve $\frac{d^3y}{dt^3} = 6$ with initial/boundary values $\quad y(t{=}0) = 5, \quad \dot{y}(t{=}0) = 0, \quad y(t{=}3) = 50.$

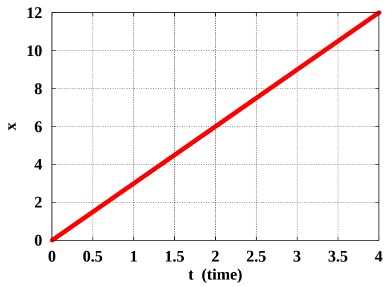
Result: $\quad y(t) = \boxed{\qquad\qquad}$

5.17 ♣ Geometric interpretations of integrals. (Section 1.7).

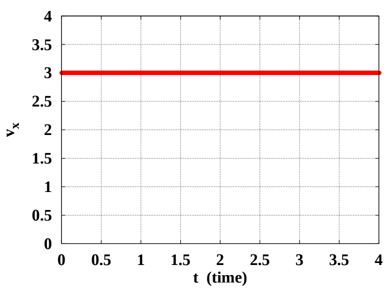
Complete the missing analytical statements and graph the missing functions.

Hint: Synthesize information from each vertical column below. Constants of integration can be deduced from graphs. For example, for the 1^{st} column, start at the bottom with $\frac{d^2x}{dt^2} = 0$ and work upward to determine $\frac{dx}{dt}$ and then $x(t)$.

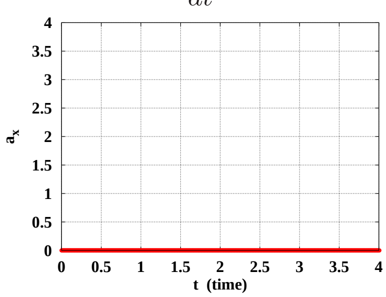
$x = \boxed{\quad}$



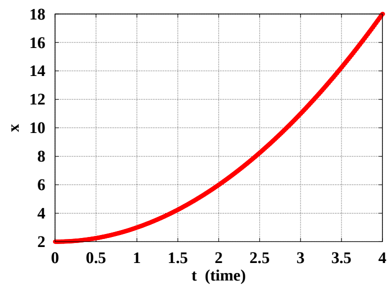
$v_x \triangleq \frac{dx}{dt} = \boxed{\quad}$



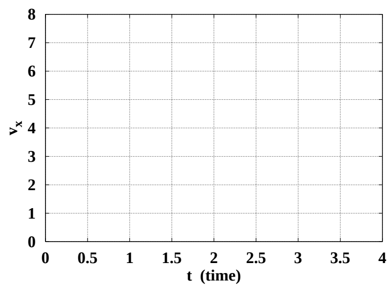
$a_x \triangleq \frac{d^2x}{dt^2} = 0$



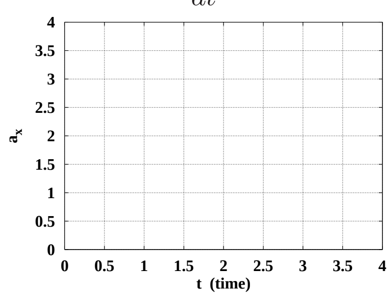
$x = \boxed{\quad} + \boxed{\quad}$



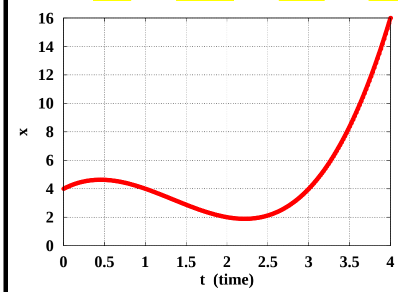
$v_x \triangleq \frac{dx}{dt} = \boxed{\quad}$



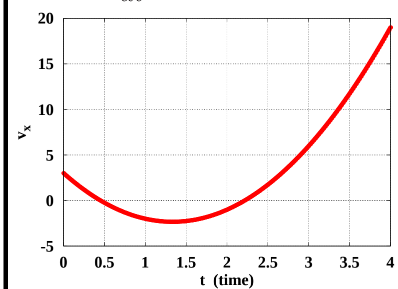
$a_x \triangleq \frac{d^2x}{dt^2} = 2$



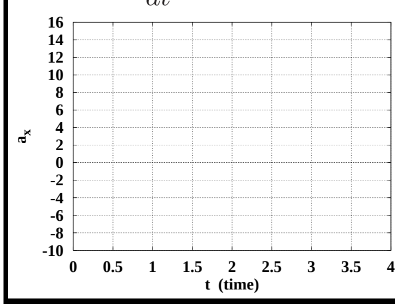
$x = \boxed{\quad} - \boxed{\quad} + \boxed{\quad} + \boxed{\quad}$



$v_x \triangleq \frac{dx}{dt} = 3\,t^2 - 8\,t + 3$



$a_x \triangleq \frac{d^2x}{dt^2} = \boxed{\quad} - \boxed{\quad}$



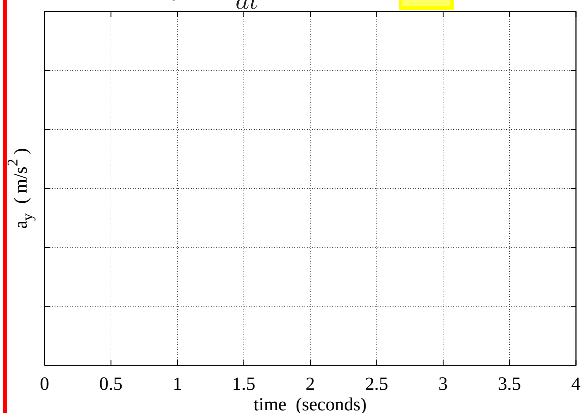
5.19 ♣ FE/EIT Review: Graphing $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$ for a sky-diver and rocket-sled.

Complete the missing statements, axes values, and graphs. Use Earth's gravitational acceleration $g \approx 10\ \frac{\text{m}}{\text{s}^2}$.

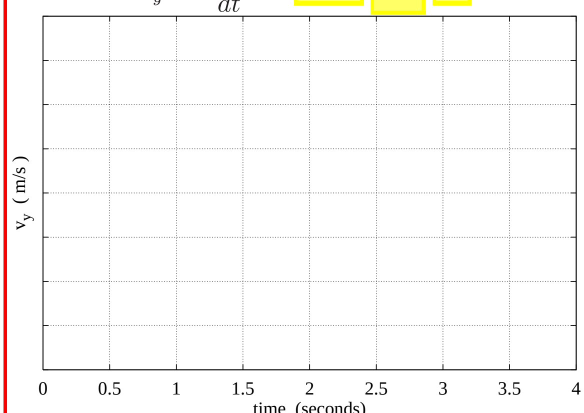
A sky-diver free-falls for 4 seconds after leaving a stationary helicopter from a height $y = 200$ m above Earth (y is positive-upward).

The only relevant force is Earth's gravity.

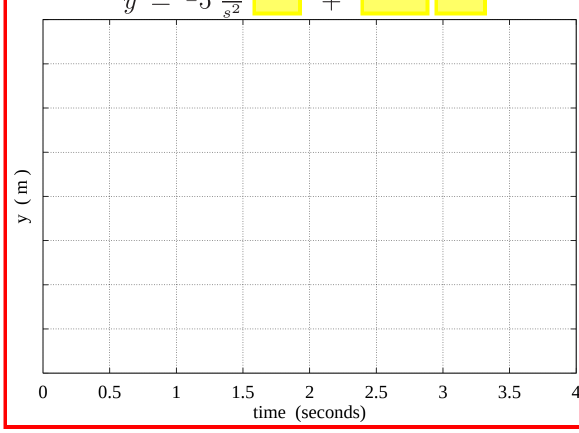
$$a_y \triangleq \frac{d^2y}{dt^2} = \square\ \square$$



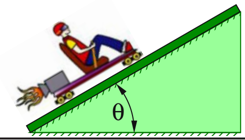
$$v_y \triangleq \frac{dy}{dt} = \square\ \square\ \square$$



$$y = -5\ \tfrac{m}{s^2}\ \square + \square\ \square$$

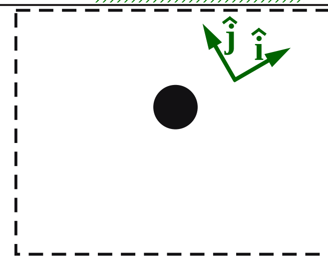


A rocket-sled of mass m is thrust along smooth inclined rails with time-varying force F_T.
The variable x measure's the sled's position along the rails.
Initially, $x = 0$ and $\dot{x} = 0$.



FBD. **Draw forces**

Below: Form $\vec{\mathbf{F}}_{\text{Net}}$ and then set $\vec{\mathbf{F}}_{\text{Net}} = m\,\vec{\mathbf{a}}$.
Use symbols m, g, F_T, F_N, θ.



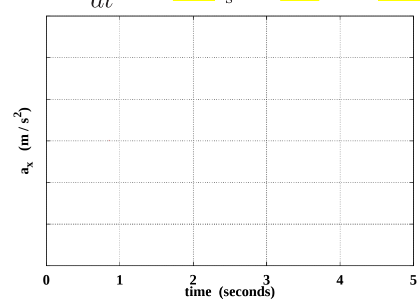
$$\square\ \hat{\mathbf{i}} + \square\ \hat{\mathbf{j}} = m\ \square\ \hat{\mathbf{i}}$$

$$\hat{\mathbf{i}}:\ \square = \square \quad \Rightarrow \quad \frac{d^2x}{dt^2} = \square - \square$$

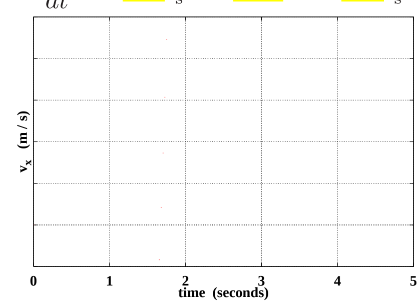
$$\hat{\mathbf{j}}:\ \square = \square \quad \Rightarrow \quad F_N = \square$$

Use $\theta = 30°$, $m = 100$ kg, $F_T = 600\ \frac{\text{N}}{\text{s}} * t$ for the following.

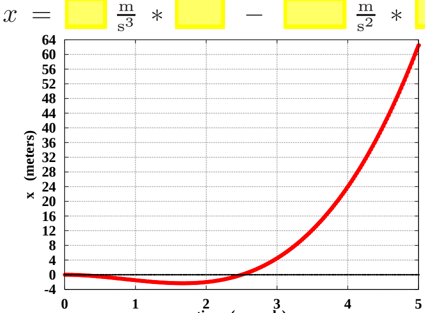
$$a_x \triangleq \frac{dv_x}{dt} = \square\ \tfrac{\text{m}}{\text{s}^3} * \square - \square\ \tfrac{\text{m}}{\text{s}^2}$$



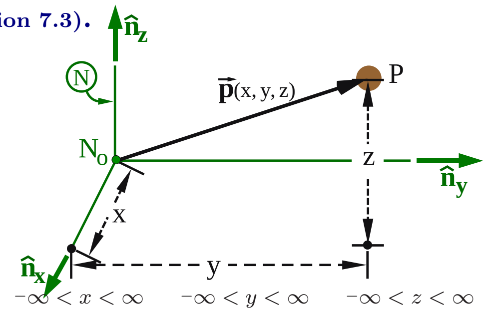
$$v_x \triangleq \frac{dx}{dt} = \square\ \tfrac{\text{m}}{\text{s}^3} * \square - \square\ \tfrac{\text{m}}{\text{s}^2} * \square$$



$$x = \square\ \tfrac{\text{m}}{\text{s}^3} * \square - \square\ \tfrac{\text{m}}{\text{s}^2} * \square$$



5.20 ♣ Vector differentiation: Cartesian coordinates. (Section 7.3).



The figure to the right shows a baseball P moving in a reference frame N. Right-handed orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are fixed in N as shown. P's location from point N_o (a point fixed in N) can be specified with three ***coordinates***.

A ***Cartesian coordinate system*** locates P with coordinates x, y, z (the $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ measures of P's position from N_o).

- Determine the time-derivative in N of $\widehat{\mathbf{n}}_x$ and briefly justify your answer.

Result: $\frac{{}^N\!d\,\widehat{\mathbf{n}}_x}{dt} = \boxed{}$ The magnitude of $\widehat{\mathbf{n}}_x$ **does/does not** change because $|\widehat{\mathbf{n}}_x| = \boxed{}$.
The direction of $\widehat{\mathbf{n}}_x$ **does/does not** change in reference frame N.

- Express P's position from N_o in terms of x, y, z, $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$. Knowing x, y, z depend on time t, and using the definition of vector derivative [equation (7.4)] and the product rule for differentiation, determine the time-derivative in N of $\vec{\mathbf{p}}$ and express it in terms of $\dot{x}$, $\dot{y}$, $\dot{z}$, and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.

Result:
$$\vec{\mathbf{p}} = \boxed{} + y\,\widehat{\mathbf{n}}_y + z\,\widehat{\mathbf{n}}_z \qquad \frac{{}^N\!d\vec{\mathbf{p}}}{dt} = \dot{x}\,\widehat{\mathbf{n}}_x + \boxed{}\,\widehat{\mathbf{n}}_y + \boxed{}\,\widehat{\mathbf{n}}_z$$

5.21 ♣ Vector differentiation and reference frames. (Section 7.3).

The following vectors are expressed in terms of orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and t time. Circle the vectors that can be differentiated without consideration of a reference frame.

$$\vec{\mathbf{0}} \qquad 2\,\widehat{\mathbf{a}}_x + 4\,\widehat{\mathbf{a}}_y \qquad 2\,\widehat{\mathbf{a}}_x + t\,\widehat{\mathbf{a}}_y$$
$$\widehat{\mathbf{a}}_x \qquad 2\,\widehat{\mathbf{a}}_x + 4\,\widehat{\mathbf{a}}_y + 6\,\widehat{\mathbf{a}}_z \qquad 2\,\widehat{\mathbf{a}}_x + t\,\widehat{\mathbf{a}}_y + \sin(t)\,\widehat{\mathbf{a}}_z$$

5.22 ♣ Textbook/Internet definitions of vector differentiation. (Section 7.3).

A vector has magnitude and direction. The change of a vector's magnitude relates to scalar differentiation. The change of a vector's direction depends on a **reference frame** (or rigid basis). The first notation that explicitly showed dependence of a ***vector derivative*** on a ***reference frame*** was by the dynamicist Thomas Kane in 1950. Kane taught that a mathematical ***definition*** should:

- Involve ingredients that themselves are reasonably understood and/or defined.
  In other words, the definition is comprehensible to the intended audience.
- Be useful for directly or indirectly proving all other related properties.

Report a definition for the derivative of a vector from a textbook (e.g., undergraduate/graduate physics or engineering textbook) and/or from the Internet, and determine if both the ***definition*** and ***notation*** clearly show that a vector's derivative depends on **reference frame** (or rigid basis).

Source (reference). List textbook or .html link	Definition. Report the defining equation/property.	Does notation explicitly show dependence on reference frame?
		Yes/No