

# Chapter 24



## MIPSI: Classic particle pendulum

The pendulum to the right consists of a small heavy metallic ball tied to a light cable which is connected to the ceiling. This chapter investigates this system with the MIPSI process (**M**odeling, **I**dentifiers, **P**hysics, **S**implify/solve, **I**nterpret/design) and shows how to use various methods to form its equations of motion.

Newton's law $\vec{F} = m\vec{a}$	Angular momentum principle
Euler's rigid body equation	Power/energy-rate principle
Conservation of mechanical energy	Lagrange's method
<b>MG road-maps</b> (see Section 21.1.8)	



Reproduction of Foucault pendulum. Pantheon, Paris France (Andrew Schmidt).

### 24.1 Modeling the classic particle pendulum

A good **model** is a reasonably accurate simplified representation of a complex system. Capturing a system's essence and extracting its meaningful parts requires good judgement and is essential to analysis and design – yet is difficult to teach. Some assumptions made when modeling a pendulum are:

1. **The support is rigid (inflexible) and the cable is rigid and does not break.**

Analyzing a flexible cable that elongates/vibrates (see Homework 7.12) or a cable that supports tension but not compression is significantly more difficult than analyzing a straight inextensible cable.<sup>1</sup>

2. **The cable is massless (significantly lighter than the objects it supports).**

One indicator of this assumption's validity is whether or not the cable's kinetic energy is substantially smaller than the attached particle's kinetic energy. A small mass or moment of inertia **cannot** be ignored if it is associated with large kinetic energy.<sup>2</sup> Kinetic energy can help for modeling a massive spring by determining how to lump some of the spring's mass with each body attached to the spring.

3. **The massive body can be modeled as a particle (body is small and dense).**

This assumption allows replacement of forces (e.g., gravity) on the object by a simpler equivalent set. This assumption seems reasonable if the body's orientation is not of interest and the body's rotational kinetic energy is substantially smaller than its translational kinetic energy.

4. **The cable has a simple rotation relative to the room (simple angular velocity).**

This implies the system's motion can be described with **one** dependent variable (e.g.,  $\theta$ ), and is valid if motion occurs over a short period of time (less than a few hours). Over longer periods of time, the Earth's rotation causes the particle to move out of plane (Foucault pendulum).

<sup>1</sup>One can test the inextensible-cable-assumption by numerical simulation of a pendulum with an extensible cable and observing that, as the cable's stiffness is increased, the solution approaches the rigid-cable solution. Note: Computer time to numerically simulate motion increases with cable stiffness. When the period of the cable's extensional oscillations are much shorter than the pendulum's period, the system is said to have **stiff differential equations**.

<sup>2</sup>For example, it may be **unreasonable** to ignore the small moment of inertia associated with a rapidly spinning small motor that is geared to a slowly spinning large object. Rotational kinetic energy scales with  $I\omega^2$ , so a rapidly spinning small motor may have more kinetic energy than a slowly spinning massive object.

5. **The Earth is a Newtonian reference frame.**

**Newton's law of motion**  $\vec{F} = m\vec{a}$  requires a non-accelerating and non-rotating reference frame. Although Earth is rotating (daily around its axis and yearly about the sun), the acceleration associated with these motions is **assumed** to be insignificantly small. Foucault showed the Earth's rotation has a substantial effect on daily motions of a simple pendulum.

6. **Earth's gravitational attraction can be approximated as a uniform field.**

In reality, gravitational forces vary as an inverse-square law. Hence, Earth's gravitational forces on the pendulum decrease as the pendulum swings up (away from Earth) and increase as it swings down (closer to Earth). Note: Uniform gravity is not used for analyzing satellite orientation (in motion for years).

7. **Other than Earth, gravitational forces are negligible.**

Other massive objects (the moon, sun, people, etc.) have negligible gravitational attraction.

An example helps validate this assumption. The magnitude of gravitational force between two lead spheres of radius 1 ft (30 cm) and mass 3035 lbm (1377 kg) whose centers are only 3 ft (91 cm) apart (closest point is 1 foot apart) is 0.000034 lbf (0.00015 N),  $\approx 90,000,000$  times smaller than Earth's gravitational force exerted on the sphere.

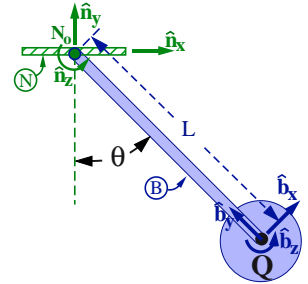
8. **Many forces (aerodynamic, friction, magnetic, electrostatic) are negligible.**

Note: Include additional forces to improve model validity (e.g., air resistance affects long-term behavior).

## 24.2 Identifiers for the classic particle pendulum

The schematic to the right shows a light (massless) inextensible cable  $B$  with a particle  $Q$  attached to its distal end. The cable is connected to the ceiling  $N$  (a Newtonian reference frame) at point  $N_o$ .

Right-handed orthogonal unit vectors  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  are fixed in  $N$  with  $\hat{n}_x$  horizontally-right,  $\hat{n}_y$  vertically-upward, and  $\hat{n}_z$  parallel to  $B$ 's axis of rotation in  $N$ . Unit vectors  $\hat{b}_x, \hat{b}_y, \hat{b}_z$  are fixed in  $B$  with  $\hat{b}_y$  from  $Q$  to  $N_o$  and  $\hat{b}_z = \hat{n}_z$ .



Quantity	Identifier	Type	Value
Earth's gravitational constant	$g$	Constant	9.8 m/s <sup>2</sup>
Mass of $Q$	$m$	Constant	2 kg
Length of $B$	$L$	Constant	1.0 m
Angle from $\hat{n}_y$ to $\hat{b}_y$ with $+\hat{n}_z$ sense	$\theta$	Variable	60° (initial)
Tension in cable	$T$	Variable	

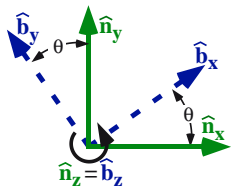
## 24.3 Physics: Equations of motion of the classic particle pendulum

There are many methods for forming equations of motion and each method requires force and kinematic analysis so these quantities are summarized below. A few of methods for forming this system's equations of motion are presented in the next sections. Section 21.1.8 provides an efficient **MG road-map** for this problem.

• **Rotational kinematics: Rotation matrix, angular velocity, angular acceleration.**

The rotation matrix  ${}^bR^n$  relates  $\hat{b}_x, \hat{b}_y, \hat{b}_z$  to  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  and is formed by drawing the unit vectors in a suggestive way as shown below and resolving them into components using  $\sin(\theta)$  and  $\cos(\theta)$ .

${}^N\vec{\omega}^B$  ( $B$ 's angular velocity in  $N$ ) is simple since  $\hat{b}_z$  is fixed in both  $B$  and  $N$  so it is formed using the right-hand rule and viewing how  $\theta$  increases.  ${}^N\vec{\alpha}^B$  is formed by its definition as shown below.



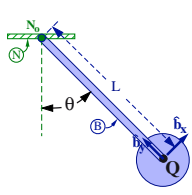
${}^bR^n$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{b}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\hat{b}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\hat{b}_z$	0	0	1

$${}^N\vec{\omega}^B = \dot{\theta} \hat{b}_z$$

$${}^N\vec{\alpha}^B \triangleq \frac{{}^N d {}^N\vec{\omega}^B}{dt} = \ddot{\theta} \hat{b}_z \quad (1)$$

• **Translational kinematics: Position, velocity, and acceleration.**

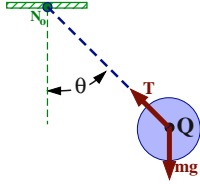
The schematic below-left shows  $Q$ 's position from  $N_o$  is  ${}^{N_o}\vec{\mathbf{r}}^Q = -L\hat{\mathbf{b}}_y$ .  
 $Q$ 's velocity and acceleration in  $N$  are calculated by their definitions as



$${}^N\vec{\mathbf{v}}^Q \triangleq \frac{{}^N d {}^{N_o}\vec{\mathbf{r}}^Q}{dt} = \frac{{}^N d (-L\hat{\mathbf{b}}_y)}{dt} = \frac{{}^B d (-L\hat{\mathbf{b}}_y)}{dt} + {}^N\vec{\boldsymbol{\omega}}^B \times (-L\hat{\mathbf{b}}_y) = \dot{\theta} L \hat{\mathbf{b}}_x \quad (2)$$

$${}^N\vec{\mathbf{a}}^Q \triangleq \frac{{}^N d {}^N\vec{\mathbf{v}}^Q}{dt} \stackrel{(2)}{=} \frac{{}^N d (\dot{\theta} L \hat{\mathbf{b}}_x)}{dt} = \frac{{}^B d (\dot{\theta} L \hat{\mathbf{b}}_x)}{dt} + {}^N\vec{\boldsymbol{\omega}}^B \times (\dot{\theta} L \hat{\mathbf{b}}_x) = \ddot{\theta} L \hat{\mathbf{b}}_x + \dot{\theta}^2 L \hat{\mathbf{b}}_y \quad (3)$$

• **Forces (FBD):**



The resultant force on particle  $Q$  is 
$$\vec{\mathbf{F}}^Q = \underbrace{T \hat{\mathbf{b}}_y}_{\text{tension}} - \underbrace{m g \hat{\mathbf{n}}_y}_{\text{gravity}} \quad (4)$$

Note: The only force a massless inextensible cable can exert on  $Q$  is a tensile force in the  $\hat{\mathbf{b}}_y$  direction. The cable's inability to exert force on  $Q$  in the  $\hat{\mathbf{b}}_x$  direction is a consequence of the Newton/Euler laws and a **free-body analysis (FBD)** of the cable.

### 24.3.1 $\vec{\mathbf{F}} = m \vec{\mathbf{a}}$ for the particle pendulum

Substituting equations (3) and (4) into  $\vec{\mathbf{F}} = m \vec{\mathbf{a}}$  gives the **vector** equation of motion:

$$\underbrace{T \hat{\mathbf{b}}_y - m g \hat{\mathbf{n}}_y}_{\vec{\mathbf{F}}^Q} = \underbrace{m (L \ddot{\theta} \hat{\mathbf{b}}_x + L \dot{\theta}^2 \hat{\mathbf{b}}_y)}_{m {}^N\vec{\mathbf{a}}^Q}$$

To form a **scalar** equation from a vector equation, one must dot-multiply with a vector. Although  $\hat{\mathbf{n}}_x$  and  $\hat{\mathbf{n}}_y$  may seem like good choices, the more efficient choice is  $\hat{\mathbf{b}}_x$  and  $\hat{\mathbf{b}}_y$ .

**Dot-multiplication of  $\vec{\mathbf{F}} = m \vec{\mathbf{a}}$  with  $\hat{\mathbf{n}}_x$  and  $\hat{\mathbf{n}}_y$**  – a set of equations **coupled** in  $\ddot{\theta}$  and  $T$ .

Shown below is the solution process for  $T$  and  $\ddot{\theta}$  for dot-multiplication with  $\hat{\mathbf{n}}_x$  and  $\hat{\mathbf{n}}_y$ . In general, **free-body analyses** result in equations that are **coupled** in force scalars (e.g.,  $T$ ) and acceleration scalars (e.g.,  $\ddot{\theta}$ ).

$$\begin{aligned} -T \sin(\theta) &\stackrel{(1)}{=} m [L \ddot{\theta} \cos(\theta) - L \dot{\theta}^2 \sin(\theta)] \\ T \cos(\theta) - m g &\stackrel{(1)}{=} m [L \ddot{\theta} \sin(\theta) + L \dot{\theta}^2 \cos(\theta)] \end{aligned} \quad \Rightarrow \quad \begin{aligned} \ddot{\theta} &= \frac{-g}{L} \sin(\theta) \\ T &= m g \cos(\theta) + m L \dot{\theta}^2 \end{aligned}$$

**Dot-multiplication of  $\vec{\mathbf{F}} = m \vec{\mathbf{a}}$  with  $\hat{\mathbf{b}}_x$  and  $\hat{\mathbf{b}}_y$**  – a set of equations **uncoupled** in  $\ddot{\theta}$  and  $T$ .

Shown below is the **simpler** solution process for  $T$  and  $\ddot{\theta}$  for dot-multiplication with  $\hat{\mathbf{b}}_x$  and  $\hat{\mathbf{b}}_y$ .

Note: Kane's method cleverly chooses vectors for dot-multiplication that simplify the resulting scalar equations.

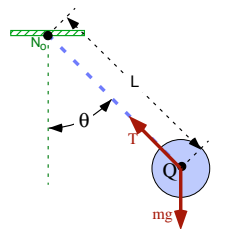
$$\begin{aligned} \text{For } \hat{\mathbf{b}}_x: \quad & -m g \sin \theta = m L \ddot{\theta} & \text{For } \hat{\mathbf{b}}_y: \quad & T - m g \cos(\theta) = m L \dot{\theta}^2 \\ \text{Solve for } \ddot{\theta}: \quad & \ddot{\theta} = \frac{-g}{L} \sin(\theta) & \text{Solve for } T: \quad & T = m g \cos(\theta) + m L \dot{\theta}^2 \end{aligned}$$

### 24.3.2 Angular momentum principle for the particle pendulum

To form equations of motion with the **angular momentum principle**, it helps to cleverly choose the “about-point”  $N_o$  as done below (this clever choice is automated by the **MG road-map** for  $\theta$  in Section 21.1.8).

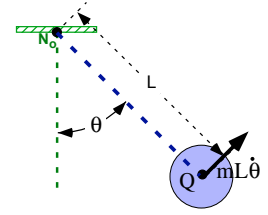
The moment of all forces on  $Q$  about  $N_o$  is calculated with the cross product of  ${}^{N_o}\vec{\mathbf{r}}^Q$  ( $Q$ 's position from  $N_o$ ) with  $\vec{\mathbf{F}}^Q$  (the resultant of all forces on  $Q$ ), as

$$\vec{\mathbf{M}}^{Q/N_o} = {}^{N_o}\vec{\mathbf{r}}^Q \times \vec{\mathbf{F}}^Q = -L \hat{\mathbf{b}}_y \times (T \hat{\mathbf{b}}_y - m g \hat{\mathbf{n}}_y) \stackrel{(1)}{=} -m g L \sin(\theta) \hat{\mathbf{b}}_z$$



$Q$ 's angular momentum about  $N_o$  in  $N$  is the cross product of  ${}^{N_o}\vec{r}^Q$  with  $m {}^N\vec{v}^Q$ .

$${}^N\vec{H}^{Q/N_o} = \underbrace{-L \hat{\mathbf{b}}_y}_{N_o\vec{r}^Q} \times \underbrace{m L \dot{\theta} \hat{\mathbf{b}}_x}_{m {}^N\vec{v}^Q} = m L^2 \dot{\theta} \hat{\mathbf{b}}_z$$

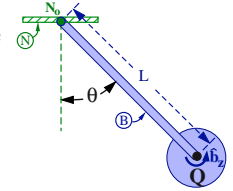


Assembling the ingredients in the angular momentum principle, one finds

$$\boxed{\vec{M}^{Q/N_o} = \frac{N_d {}^N\vec{H}^{Q/N_o}}{dt}} \Rightarrow -m g L \sin(\theta) \hat{\mathbf{b}}_z = m L^2 \ddot{\theta} \hat{\mathbf{b}}_z \Rightarrow \ddot{\theta} + \frac{g}{L} \sin(\theta) = 0$$

### 24.3.3 Euler's rigid body equation for the particle pendulum

Since  $B$  and  $Q$  can be regarded as a rigid body with planar motion in a Newtonian frame  $N$ , Euler's equation for a rigid body with a *simple angular velocity* can be used.



$$\boxed{\vec{M}_z^{B/N_o} = \underset{(2D)}{I} {}^N\vec{\alpha}^B} \Rightarrow \underbrace{-m g L \sin(\theta) \hat{\mathbf{b}}_z}_{\vec{M}_z^{B/N_o}} = \underbrace{m L^2}_{I} \underbrace{\ddot{\theta} \hat{\mathbf{b}}_z}_{{}^N\vec{\alpha}^B} \Rightarrow \ddot{\theta} + \frac{g}{L} \sin(\theta) = 0$$

- $\vec{M}_z^{B/N_o} = -m g L \sin(\theta) \hat{\mathbf{b}}_z$  is the  $\hat{\mathbf{b}}_z$  component of the moment of all forces on  $B$  about  $N_o$ , (see Section 24.3.2).
- $I$  is  $B$ 's mass moment of inertia about the line passing through  $N_o$  and parallel to  $\hat{\mathbf{b}}_z$ .  
Since  $B$ 's mass is solely particle  $Q$ ,  $I = m L^2$  ( $Q$ 's mass multiplied by the square of  $Q$ 's distance from  $N_o$ ).
- ${}^N\vec{\alpha}^B$  is  $B$ 's angular acceleration in  $N$ , which was calculated in a previous section as  ${}^N\vec{\alpha}^B = \ddot{\theta} \hat{\mathbf{b}}_z$ .

### 24.3.4 Kinetic energy for the particle pendulum

*Kinetic energy* is useful for:

- *Power/energy-rate principle* (example in Section 24.3.5)
- *Work/energy principle* described in Section 22.7.
- *Conservation of mechanical energy* (example in Section 24.3.6).
- Lagrange's equations of motion.

$$\begin{aligned} {}^N K^Q &\triangleq \frac{1}{2} m {}^N\vec{v}^Q \cdot {}^N\vec{v}^Q \\ &= \frac{1}{2} m (L \dot{\theta} \hat{\mathbf{b}}_x) \cdot (L \dot{\theta} \hat{\mathbf{b}}_x) \\ &= \frac{1}{2} m L^2 \dot{\theta}^2 \end{aligned}$$

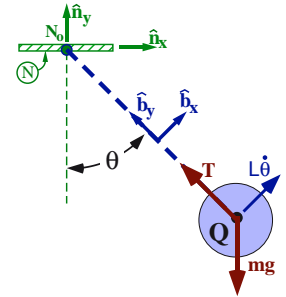
### 24.3.5 Power/energy-rate principle for the particle pendulum

The following figure shows the forces and velocity for particle  $Q$ . Since this particle-pendulum has one-degree of freedom in  $N$  (its motion is described by the **one** variable  $\dot{\theta}$ ), the *power/energy-rate principle* described in Section 22.1 is useful for forming its one equation of motion.

The power of all forces on  $Q$  in  $N$  is defined as

$${}^N P^Q \triangleq \vec{\mathbf{F}}^Q \cdot {}^N\vec{v}^Q = (T \hat{\mathbf{b}}_y - m g \hat{\mathbf{n}}_y) \cdot (L \dot{\theta} \hat{\mathbf{b}}_x) = -m g L \sin(\theta) \dot{\theta}$$

Since tension  $T$  does **not** appear in the power of  $Q$ , it is called a *workless force* or *non-contributing force*. The fortuitous absence of tension (an unknown) is one of the major advantages of the power/energy-rate principle.



Equating power to the time-derivative of kinetic energy gives:

$$\text{Power/energy-rate principle} \quad \boxed{{}^N P^Q = \frac{d {}^N K^Q}{dt}} \quad {}^N K^Q = \frac{1}{2} m L^2 \dot{\theta}^2 \Rightarrow -m g L \sin(\theta) \dot{\theta} = m L^2 \dot{\theta} \ddot{\theta}$$

Rearranging and factoring out  $\dot{\theta}$ , noting that in general  $\dot{\theta} \neq 0$ , and dividing by  $m L^2$ , gives

$$[m L^2 \ddot{\theta} + m g L \sin(\theta)] \dot{\theta} = 0 \Rightarrow m L^2 \ddot{\theta} + m g L \sin(\theta) = 0 \Rightarrow \ddot{\theta} + \frac{g}{L} \sin(\theta) = 0$$

Note: Multiplying the power/energy-rate equation by the differential  $dt$  and integrating the left-hand side with respect to  $d\theta$  and the right-hand side with respect to  $d\dot{\theta}$  leads to *conservation of mechanical energy*  $m g L \cos(\theta) = \frac{1}{2} m L^2 \dot{\theta}^2$ .

The **power/energy-rate principle** in Section 22.1 is useful for several reasons:

- It forms an equation of motion for systems with **one**-degree of freedom.
- It tells us that gravity  $g$  affects the  $Q$ 's motion in  $N$ , but tension  $T$  does not.
- Multiplying both sides of the power/energy-rate equation by the differential  $dt$  and integrating the left-hand side with respect to  $d\theta$  and the right-hand side with respect to  $d\dot{\theta}$  produces a **work/energy principle** where  $C$  is an arbitrary constant of integration. Note: Since work is a function of only configuration, a potential energy exists and this integral represents **conservation of mechanical energy**.

$$m g L \cos(\theta) + C = \frac{1}{2} m L^2 \dot{\theta}^2 \quad \text{or} \quad C = \frac{1}{2} m L^2 \dot{\theta}^2 + -m g L \cos(\theta)$$

### 24.3.6 Conservation of mechanical energy for the particle pendulum

**Conservation of mechanical energy** described in Section 23.2 is the time-integral of the **power/energy-rate principle** and relates the system's kinetic and potential energies to an arbitrary constant (e.g., Constant = 0) as

$$\boxed{K + U = \text{Constant}} \quad \Rightarrow \quad \frac{1}{2} m L^2 \dot{\theta}^2 + -m g L \cos(\theta) = 0$$

The time-derivative of **conservation of mechanical energy** produces **one** time-dependent equation of motion (namely the **power/energy-rate principle**) as

$$m L^2 \ddot{\theta} + m g L \sin(\theta) \dot{\theta} = 0$$

Rearranging and factoring out  $\dot{\theta}$ , noting that in general  $\dot{\theta} \neq 0$ , and dividing by  $m L^2$ , gives

$$[m L^2 \ddot{\theta} + m g L \sin(\theta)] \dot{\theta} = 0 \quad \Rightarrow \quad m L^2 \ddot{\theta} + m g L \sin(\theta) = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{L} \sin(\theta) = 0$$

### 24.3.7 Lagrange's method for the particle pendulum

As described in Section 20.7, **Lagrange's equations of the second kind** relate partial derivatives of potential energy  $U$  and kinetic energy  $K$  as shown below [from equation (20.7)].

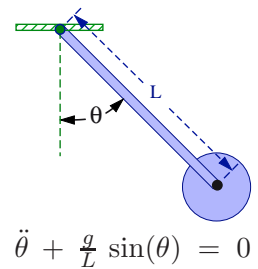
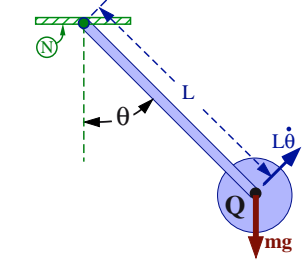
$$\boxed{-\frac{\partial U}{\partial \theta} \stackrel{(20.7)}{=} \frac{d}{dt} \frac{\partial K}{\partial \dot{\theta}} - \frac{\partial K}{\partial \theta}} \quad \begin{array}{l} U = m g \text{ height} = -m g L \cos(\theta) \\ K = \frac{1}{2} m {}^N \vec{v}^Q \cdot {}^N \vec{v}^Q = \frac{1}{2} m L^2 \dot{\theta}^2 \end{array} \quad m L^2 \ddot{\theta} + m g L \sin(\theta) = 0$$

Dividing by  $m L^2$  and rearrangement gives the classic pendulum equation  $\ddot{\theta} + \frac{g}{L} \sin(\theta) = 0$ .

## 24.4 Solution of the classic particle pendulum ODE

The ODE (ordinary differential equation) that governs the motion of the classic pendulum is nonlinear, homogeneous, constant-coefficient, and  $2^{nd}$ -order. There are a variety of methods to solve this ODE for  $\theta(t)$ , including:

- Numerical integration with a computer program.
- Analytical solution with Jacobian elliptic functions.
- Analytical solution using the small angle approximation  $\sin(\theta) \approx \theta$ .



### 24.4.1 Numerical solution of pendulum ODE via MotionGenesis and/or MATLAB®

There are **few** mechanical systems whose ODEs have analytical solutions whereas there are **many** mechanical systems whose ODEs can be solved with numerical integration (e.g., a variable-step Runge-Kutta integrator). For example, the following MotionGenesis commands numerically solve the ODE for  $\theta(t)$ .

To make MotionGenesis create a .m file for subsequent use with MATLAB®, change the last line to pendulum.m.

```
Constant g = 9.8 m/s^2;    L = 1 m           % Declare g (gravity) and L (length).
Variable theta'' = -g/L*sin(theta)          % Governing ode (ordinary differential equation).
Input theta = 60 deg, theta' = 0 deg/sec    % Initial values for theta and theta'.
Input tFinal = 6 sec, tStep = 0.02 sec      % Solve ODE from t=0 to 6 sec. Output every 0.02 sec.
OutputPlot t sec, theta deg                 % Have ODE output and plot theta vs. t.
```



### 24.4.2 Optional: Exact (closed-form) solution of the classic particle pendulum ODE

The simple particle pendulum is one of the **few** mechanical systems whose motion are governed by a **nonlinear** ODE which has a known analytical (closed-form) solution. The exact analytical solution for its ODE is a **Jacobian elliptic function** with an pendulum oscillation period  $\tau_{\text{period}}$  that depends on  $K(k)$ , the elliptic integral of the first kind with modulus  $k = \sin(\frac{\theta_0}{2})$  [ $\theta_0$  is initial value of  $\theta$ ].

**Exact:**  $\tau_{\text{period}} = 4 K(k) \sqrt{\frac{L}{g}}$       **Approximate:**  $\tau_{\text{period}} \approx \frac{2\pi}{\omega}$  where  $\omega = \sqrt{\frac{g}{L}} (1 - k^2)^{0.25 \cos(\frac{\theta_0}{2})^{0.125}}$

Note: This approximation from [29] for  $\tau_{\text{period}}$  is accurate to 1% for angles up to  $177^\circ$  and computable with a simple calculator.

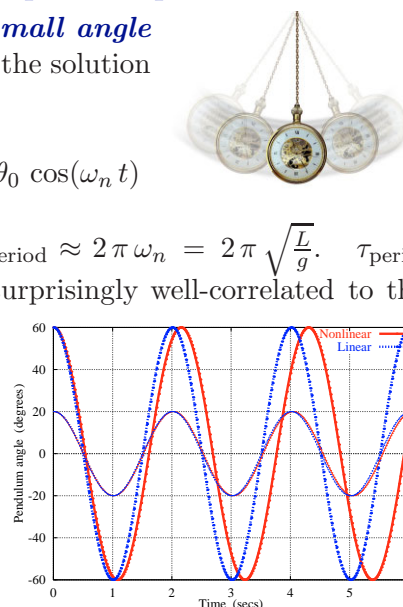
### 24.4.3 Simplification and analytical solution of the classic particle pendulum ODE

One way to approximate the pendulum's **nonlinear** ODE is with the **small angle approximation**  $\sin(\theta) \approx \theta$ , which makes the ODE **linear** and allows the solution to be written in terms of  $\omega_n \triangleq \sqrt{\frac{g}{L}}$  and the initial values  $\theta_0$  and  $\dot{\theta}_0$ .

$$\ddot{\theta} + \frac{g}{L} \sin(\theta) = 0 \quad \xRightarrow{\text{small angle}} \quad \ddot{\theta} + \frac{g}{L} \theta \approx 0 \quad \xRightarrow{\text{solve}} \quad \theta(t) \approx \frac{\dot{\theta}_0}{\omega_n} \sin(\omega_n t) + \theta_0 \cos(\omega_n t)$$

The period of oscillations of the approximate analytical solution is  $\tau_{\text{period}} \approx 2\pi \omega_n = 2\pi \sqrt{\frac{L}{g}}$ .  $\tau_{\text{period}}$  can also be experimentally determined with a real pendulum and is surprisingly well-correlated to this analytically simple model and small-angle approximation for  $\tau_{\text{period}}$ .<sup>3</sup>

**Linear and nonlinear solution:** In addition to correlating the period, it is helpful to compare the motion predicted by the linear differential equation with that predicted by the exact solution of the nonlinear differential equation as is done in the figure to the right. For small initial angles, e.g.,  $\theta_0 = 20^\circ$ , the linear and nonlinear differential equations predict similar motions. For larger initial angles of  $\theta$ , e.g.,  $\theta_0 = 60^\circ$ , the motion differ more, but is still surprisingly similar even though  $\theta$  is large ( $|\theta| > 1$  rad).



## 24.5 Interpretation of results for the classic particle pendulum

The investigation of the classic particle pendulum leads to several conclusions:

- There are many ways to form equations of motion for a simple pendulum.
- The motion of the pendulum does not depend on the mass of the object.
- For small angles, the period of oscillation does **not** depend on the initial value of  $\theta$ .
- For large initial angles (e.g.,  $\theta_0 = 1$  rad), the motion predicted by the linear (small angle) ODE differs surprisingly little from the motion predicted by the full nonlinear ODE.



<sup>3</sup>The experimental determination of a pendulum's period requires a string, a tape measure, and a stop watch. It is easy to time the period of a one meter long pendulum and compare it with the analytical period of  $\tau_{\text{period}} = 2\pi \sqrt{1/9.8} = 2.007$  sec. Other interesting results for this simple pendulum were reported in [56] by Vassar College Physicist Cindy Schwarz.