11.1 ♣ Velocity variables and degrees of freedom (Section 11.1).

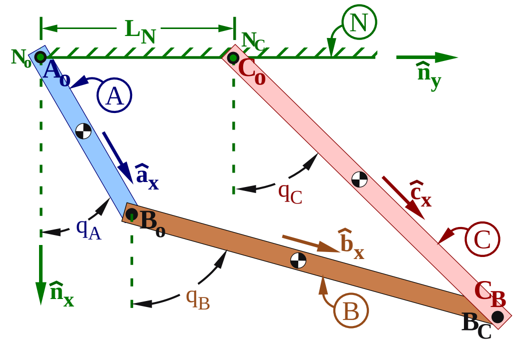
Determine the minimum number of unknown ***velocity variables*** necessary to characterize the motion of the following systems in a reference frame N. Regard Q as a free-flying particle and A as a rigid body that is **free** to translate and rotate in 3D-space. Choose and **define** velocity variables that suffice to describe the motion (Note: The choice of velocity variables is **not unique**).

Optional: Sketch each system with names for each point/body.

System (Q, B, or A and B)	Degrees of freedom	Choice of velocity variables
Free-flying particle Q.	3	$v_x \;\; v_y \;\; v_z$ ${}^N\vec{\mathbf{v}}^Q = v_x\,\widehat{\mathbf{n}}_\mathrm{x} + v_y\,\widehat{\mathbf{n}}_\mathrm{y} + v_z\,\widehat{\mathbf{n}}_\mathrm{z}$
Particle Q moving in a slot (slot is parallel to a unit vector $\vec{\mathbf{n}}$).	1	v ${}^N\vec{\mathbf{v}}^Q = v\,\vec{\mathbf{n}}$
Free-flying rigid body B.	6	$\omega_x \;\; \omega_y \;\; \omega_z \;\; v_x \;\; v_y \;\; v_z$ ${}^N\vec{\boldsymbol{\omega}}^B = \omega_x\,\widehat{\mathbf{b}}_\mathrm{x} + \omega_y\,\widehat{\mathbf{b}}_\mathrm{y} + \omega_z\,\widehat{\mathbf{b}}_\mathrm{z}$ ${}^N\vec{\mathbf{v}}^{B_o} = v_x\,\widehat{\mathbf{n}}_\mathrm{x} + v_y\,\widehat{\mathbf{n}}_\mathrm{y} + v_z\,\widehat{\mathbf{n}}_\mathrm{z}$
Rigid body B connected to rigid body A by a revolute joint. (A connects to B at point A_B of A)	7	$\omega_x \;\; \omega_y \;\; \omega_z \;\; v_x \;\; v_y \;\; v_z \;\; \omega_B$ ${}^N\vec{\boldsymbol{\omega}}^A = \omega_x\,\widehat{\mathbf{a}}_\mathrm{x} + \omega_y\,\widehat{\mathbf{a}}_\mathrm{y} + \omega_z\,\widehat{\mathbf{a}}_\mathrm{z}$ ${}^N\vec{\mathbf{v}}^{A_B} = v_x\,\widehat{\mathbf{a}}_\mathrm{x} + v_y\,\widehat{\mathbf{a}}_\mathrm{y} + v_z\,\widehat{\mathbf{a}}_\mathrm{z}$ ${}^A\vec{\boldsymbol{\omega}}^B = \omega_B\,\widehat{\boldsymbol{\lambda}}$
Rigid body B connected to rigid body A by a rigid joint.	6	$\omega_x \;\; \omega_y \;\; \omega_z \;\; v_x \;\; v_y \;\; v_z$ ${}^N\vec{\boldsymbol{\omega}}^A = \omega_x\,\widehat{\mathbf{a}}_\mathrm{x} + \omega_y\,\widehat{\mathbf{a}}_\mathrm{y} + \omega_z\,\widehat{\mathbf{a}}_\mathrm{z}$ ${}^N\vec{\mathbf{v}}^{A_B} = v_x\,\widehat{\mathbf{n}}_\mathrm{x} + v_y\,\widehat{\mathbf{n}}_\mathrm{y} + v_z\,\widehat{\mathbf{n}}_\mathrm{z}$
Rigid body B connected to A by a ball-and-socket joint.	9	$\omega_x \;\; \omega_y \;\; \omega_z \;\; v_x \;\; v_y \;\; v_z \;\; \omega_x^B \;\; \omega_y^B \;\; \omega_z^B$ ${}^N\vec{\boldsymbol{\omega}}^A = \omega_x\,\widehat{\mathbf{a}}_\mathrm{x} + \omega_y\,\widehat{\mathbf{a}}_\mathrm{y} + \omega_z\,\widehat{\mathbf{a}}_\mathrm{z}$ ${}^N\vec{\mathbf{v}}^{A_B} = v_x\,\widehat{\mathbf{a}}_\mathrm{x} + v_y\,\widehat{\mathbf{a}}_\mathrm{y} + v_z\,\widehat{\mathbf{a}}_\mathrm{z}$ ${}^A\vec{\boldsymbol{\omega}}^B = \omega_x^B\,\widehat{\mathbf{b}}_\mathrm{x} + \omega_y^B\,\widehat{\mathbf{b}}_\mathrm{y} + \omega_z^B\,\widehat{\mathbf{b}}_\mathrm{z}$
Rigid body B connected to A by a revolute angular velocity motor. (A revolute angular velocity motor ***specifies*** B's angular velocity in A)	6	$\omega_x \;\; \omega_y \;\; \omega_z \;\; v_x \;\; v_y \;\; v_z$ ${}^N\vec{\boldsymbol{\omega}}^A = \omega_x\,\widehat{\mathbf{a}}_\mathrm{x} + \omega_y\,\widehat{\mathbf{a}}_\mathrm{y} + \omega_z\,\widehat{\mathbf{a}}_\mathrm{z}$ ${}^N\vec{\mathbf{v}}^{A_B} = v_x\,\widehat{\mathbf{n}}_\mathrm{x} + v_y\,\widehat{\mathbf{n}}_\mathrm{y} + v_z\,\widehat{\mathbf{n}}_\mathrm{z}$
Rectangular box B sliding on a flat rigid surface fixed in N. (B contacts N at a single vertex B_o of B)	5	$\omega_x \;\; \omega_y \;\; \omega_z \;\; v_x \;\; v_y$ ${}^N\vec{\mathbf{v}}^{B_o} = v_x\,\widehat{\mathbf{n}}_\mathrm{x} + v_y\,\widehat{\mathbf{n}}_\mathrm{y}$ ${}^N\vec{\boldsymbol{\omega}}^B = \omega_x\,\widehat{\mathbf{b}}_\mathrm{x} + \omega_y\,\widehat{\mathbf{b}}_\mathrm{y} + \omega_z\,\widehat{\mathbf{b}}_\mathrm{z}$
Rectangular box B sliding on a flat rigid surface fixed in N. (B contacts N on a single edge of B)	4	$\omega_y \;\; \omega_z \;\; v_x \;\; v_y$ ${}^N\vec{\mathbf{v}}^{B_o} = v_x\,\widehat{\mathbf{n}}_\mathrm{x} + v_y\,\widehat{\mathbf{n}}_\mathrm{y}$ ${}^N\vec{\boldsymbol{\omega}}^B = \omega_y\,\widehat{\mathbf{b}}_\mathrm{y} + \omega_z\,\widehat{\mathbf{b}}_\mathrm{z}$
Rectangular box B sliding on a flat rigid surface fixed in N. (B contacts N on one surface of B)	3	$\omega_z \;\; v_x \;\; v_y$ ${}^N\vec{\mathbf{v}}^{B_o} = v_x\,\widehat{\mathbf{n}}_\mathrm{x} + v_y\,\widehat{\mathbf{n}}_\mathrm{y}$ ${}^N\vec{\boldsymbol{\omega}}^B = \omega_z\,\widehat{\mathbf{b}}_\mathrm{z}$

11.2 Position variables and constraints for a four-bar linkage (four variables).

The following figure shows a planar four-bar linkage consisting of uniform rigid links A, B, and C and ground N. Link A is connected with revolute joints to N and B at points N_A and A_B, respectively. Link C is connected with revolute joints to N and B at points C_N and B_C, respectively. One way to describe the four-bar's configuration is with **three** angles q_A, q_B, q_C (described in Homework 4.17).



Another way to describe its configuration is with **four** position variables x_1, y_1, x_2, y_2 defined as

$$x_1 \triangleq \vec{\mathbf{r}}^{A_B/N_A} \cdot \hat{\mathbf{n}}_x \qquad y_1 \triangleq \vec{\mathbf{r}}^{A_B/N_A} \cdot \hat{\mathbf{n}}_y$$
$$x_2 \triangleq \vec{\mathbf{r}}^{B_C/N_A} \cdot \hat{\mathbf{n}}_x \qquad y_2 \triangleq \vec{\mathbf{r}}^{B_C/N_A} \cdot \hat{\mathbf{n}}_y$$

Create three constraints relating x_1, y_1, x_2, y_2.

Result:

$$x_1^2 + y_1^2 - L_A^2 = 0$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - L_B^2 = 0$$
$$x_2^2 + (y_2 - L_N)^2 - L_C^2 = 0$$

11.3 Position variables and constraints for a four-bar linkage (nine variables).

The following figure shows a planar four-bar linkage consisting of uniform rigid links A, B, and C and ground N. Link A is connected with revolute joints to N and B at points N_A and A_B, respectively. Link C is connected with revolute joints to N and B at points C_N and B_C, respectively. The length of links A, B, C are L_A, L_B, L_C, respectively. The distance between N_o and C_N is L_N. A point N_o of N is **coincident** with point N_A

One way to describe the four-bar's configuration is with **three** angles q_A, q_B, q_C (described in Homework 4.17). Another way to describe its configuration is with **four** position variables x_1, y_1, x_2, y_2 (described in Homework 11.2). Yet another way to describe a system's configuration is by introducing variables that separately characterize each object's configuration (which can simplify changing constraints between objects, e.g., breaking constraints due to excessive force or imposing constraints due to contact).

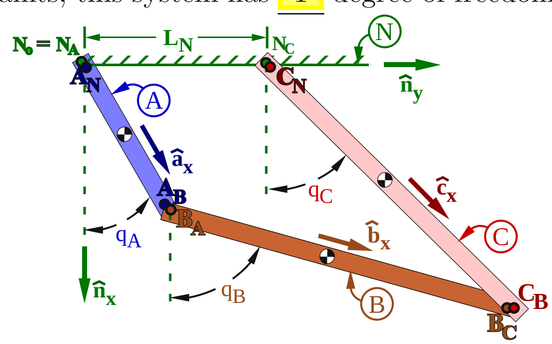
For example, for a planar four-bar linkage, the following **9** variables can be used.

$$x_A \triangleq \vec{\mathbf{r}}^{A_{cm}/N_o} \cdot \hat{\mathbf{n}}_x \qquad y_A \triangleq \vec{\mathbf{r}}^{A_{cm}/N_o} \cdot \hat{\mathbf{n}}_y \qquad q_A \triangleq \angle(\hat{\mathbf{n}}_x,\ \hat{\mathbf{a}}_x,\ {}^+\hat{\mathbf{n}}_z)$$
$$x_B \triangleq \vec{\mathbf{r}}^{B_{cm}/N_o} \cdot \hat{\mathbf{n}}_x \qquad y_B \triangleq \vec{\mathbf{r}}^{B_{cm}/N_o} \cdot \hat{\mathbf{n}}_y \qquad q_B \triangleq \angle\left(\hat{\mathbf{n}}_x,\ \hat{\mathbf{b}}_x,\ {}^+\hat{\mathbf{n}}_z\right)$$
$$x_C \triangleq \vec{\mathbf{r}}^{C_{cm}/N_o} \cdot \hat{\mathbf{n}}_x \qquad y_C \triangleq \vec{\mathbf{r}}^{C_{cm}/N_o} \cdot \hat{\mathbf{n}}_y \qquad q_C \triangleq \angle(\hat{\mathbf{n}}_x,\ \hat{\mathbf{c}}_x,\ {}^+\hat{\mathbf{n}}_z)$$

Create **8** independent scalar configuration constraints of the form $p_i = 0$ interrelating these **9** variables. With **9** variables and **8** independent scalar constraints, this system has $\boxed{1}$ degree of freedom.

Result:

$$x_A - 0.5\,L_A\cos(q_A) = 0$$
$$y_A - 0.5\,L_A\sin(q_A) = 0$$
$$x_B - x_A - 0.5\,L_A\cos(q_A) - 0.5\,L_B\cos(q_B) = 0$$
$$y_B - y_A - 0.5\,L_A\sin(q_A) - 0.5\,L_B\sin(q_B) = 0$$
$$x_C - x_B + 0.5\,L_C\cos(q_C) - 0.5\,L_B\cos(q_B) = 0$$
$$y_C - y_B + 0.5\,L_C\sin(q_C) - 0.5\,L_B\sin(q_B) = 0$$
$$0.5\,L_C\cos(q_C) - x_C = 0$$
$$L_N + 0.5\,L_C\sin(q_C) - y_C = 0$$



Note: The solution to the nonlinear equations $p_i = 0$ ($i = 1, \ldots, 8$) is not unique. For example, given $x_A = 0$, these 8 equations can be solved for the other 8 variables – where the 4-bar can hang down or be flipped up. There is more than one configuration which satisfies these 8 equations.

