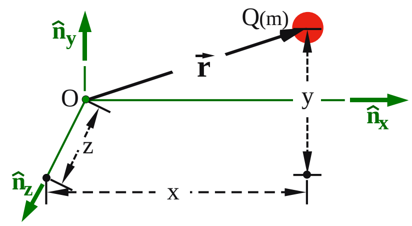
13.1 ♣ Concepts: What objects have a moment of inertia? (Section 14.1).

Consider the ***moment of inertia*** $\mathrm{I}^{S/O}_{\hat{\mathbf{u}}\,\hat{\mathbf{u}}}$ of an object S about a point O for the unit vector $\hat{\mathbf{u}}$. In general, for $\mathrm{I}^{S/O}_{\hat{\mathbf{u}}\,\hat{\mathbf{u}}}$ to be a positive real number, S should be a (circle ***all*** appropriate objects):

Real number	Matrix	Set of points	Mass center of a rigid body
Vector	Point	Reference frame	Flexible body
3D orthogonal unit basis	Particle	Rigid body	System of particles and bodies

13.2 ♣ Formulas for a particle's moments and products of inertia (Sections 14.1 and 14.2).

The figure shows a particle Q of mass m and right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$. Q's position vector from a point O is $x\,\hat{\mathbf{n}}_x + y\,\hat{\mathbf{n}}_y + z\,\hat{\mathbf{n}}_z$.



Express I_{xx} (Q's ***moment of inertia*** about O for $\hat{\mathbf{n}}_x$) in terms of some or all of m, x, y z. Similarly for I_{yy} and I_{zz}.

Express I_{xy} (Q's ***product of inertia*** about O for $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$) in terms of some or all of m, x, y z. Similarly for I_{xz} and I_{yz}.

Result:

$$\mathrm{I}_{xx} = \square\,(\square^2 + \square^2) \qquad \mathrm{I}_{yy} = \square \qquad \mathrm{I}_{zz} = \square$$

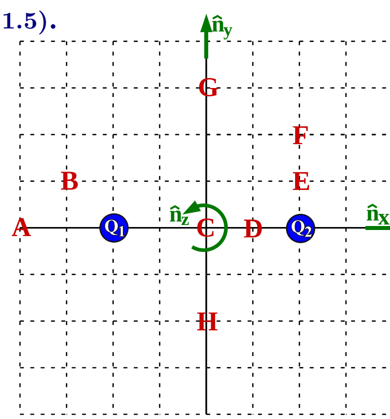
$$\mathrm{I}_{xy} = -\square\,\square\,\square \qquad \mathrm{I}_{xz} = \square \qquad \mathrm{I}_{yz} = \square$$

Circa 1895, Gibbs invented the ***inertia dyadic*** as a **convenient "suitcase"** for holding moments and products of inertia. Write Q's inertia dyadic about O in terms of $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and I_{ij} $(i, j = x, y, z)$. If needed, refer to Section 16.1.

$$\begin{aligned}\overrightarrow{\overrightarrow{\mathbf{I}}} \;=\;& \mathrm{I}_{xx}\,\hat{\mathbf{n}}_x\,\hat{\mathbf{n}}_x + \mathrm{I}_{xy}\,\hat{\mathbf{n}}_x\,\hat{\mathbf{n}}_y + \square\,\hat{\mathbf{n}}_x\,\hat{\mathbf{n}}_z \\ &+ \mathrm{I}_{xy}\,\hat{\mathbf{n}}_y\,\hat{\mathbf{n}}_x + \square\,\hat{\mathbf{n}}_y\,\hat{\mathbf{n}}_y + \square \\ &+ \square + \square + \square\end{aligned}$$

13.3 ♣ Parallel axis theorem and moments of inertia (Section 14.1.5).

The system S shown to the right consists of particles Q_1 and Q_2, each of mass m, in a plane perpendicular to the unit vector $\hat{\mathbf{n}}_z$.



The ***shift theorem*** (also called the ***parallel axis theorem***) shifts S's moment of inertia about S_{cm} (the ***mass center*** of S) for the unit vector $\hat{\mathbf{n}}_z$ to an arbitrary point P in the plane using

$$\mathrm{I}^{S/P}_{zz} = \mathrm{I}^{S/S_{\text{cm}}}_{zz} + m_S * d^2$$

where $\mathrm{I}^{S/S_{\text{cm}}}_{zz}$ is the system's moment of inertia about S_{cm} for $\hat{\mathbf{n}}_z$, m_S is the mass of S, and d is the distance from S_{cm} to P.

Use the shift theorem to estimate the order of S's moment of inertia for the lines parallel to $\hat{\mathbf{n}}_z$ that pass through points A, B, C, D, E, F, G, and H, respectively. Note: Grid lines are equally spaced.

Result:

Smallest	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	Largest

Knowing each particle has mass $m = 1$ kg and the grid lines are spaced 1 m apart, calculate S's moment of inertia about A, B, C, D, E, F, G, and H, respectively.

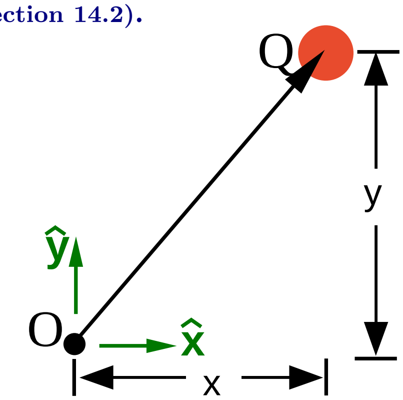
Result: (in $\text{kg}\,\text{m}^2$)

$\mathrm{I}^{S/A}_{zz}$	$\mathrm{I}^{S/B}_{zz}$	$\mathrm{I}^{S/C}_{zz}$	$\mathrm{I}^{S/D}_{zz}$	$\mathrm{I}^{S/E}_{zz}$	$\mathrm{I}^{S/F}_{zz}$	$\mathrm{I}^{S/G}_{zz}$	$\mathrm{I}^{S/H}_{zz}$
$\square$	$\square$	8	$\square$	$\square$	$\square$	$\square$	$\square$

13.4 ♣ Sign conventions for products of inertia (Section 14.2.5).

There are two sign conventions (±) for products of inertia which often lead to errors **True/False**.

13.5 ♣ Calculations: Product of inertia for a single particle (Section 14.2).



The ***product of inertia*** of a single particle Q about a point O for the $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ directions is calculated by the formula

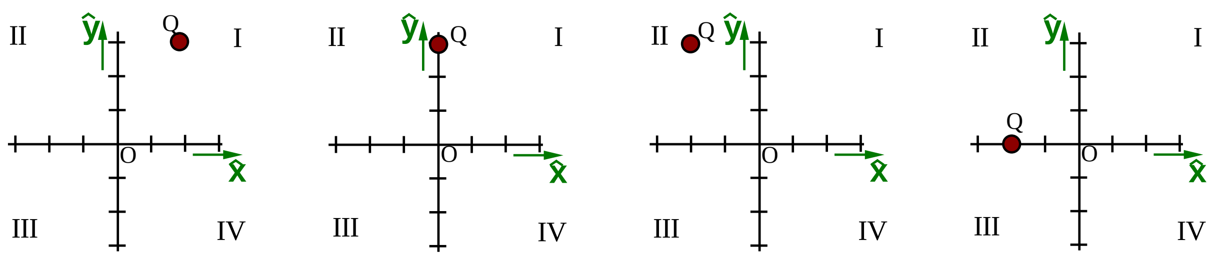
$$\boxed{\mathrm{I_{xy}} \;=\; -m\,x\,y}$$

- m is the mass of Q
- x is the $\widehat{\mathbf{x}}$ measure of Q's position from O
- y is the $\widehat{\mathbf{y}}$ measure of Q's position from O

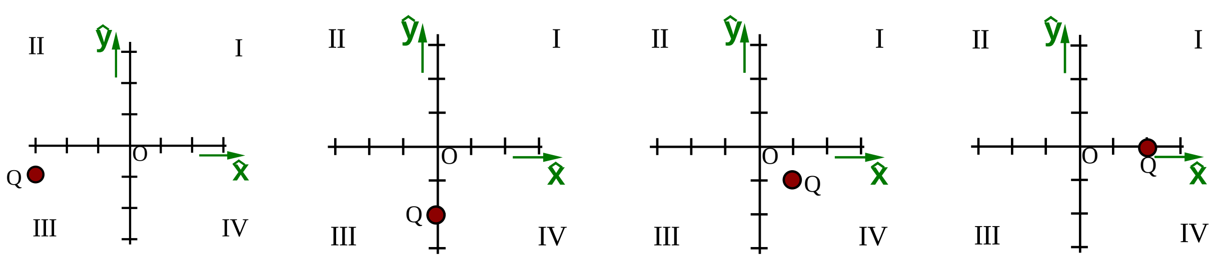
For example, if $m = 1$ kg, $x = 2$ m, and $y = 3$ m,

$$\mathrm{I_{xy}} \;=\; -(1\text{ kg})\,(2\text{ m})\,(3\text{ m}) \;=\; -6\text{ kg m}^2$$

Knowing particle Q has a mass of 1 kg and each tick-mark represents 1 m, calculate Q's ***product of inertia*** $\mathrm{I_{xy}}$ about point O for each figure below.



$\mathrm{I_{xy}} = -6$ kg m^2 $\quad$ $\mathrm{I_{xy}} = \square$ kg m^2 $\quad$ $\mathrm{I_{xy}} = \square$ kg m^2 $\quad$ $\mathrm{I_{xy}} = \square$ kg m^2



$\mathrm{I_{xy}} = -3$ kg m^2 $\quad$ $\mathrm{I_{xy}} = \square$ kg m^2 $\quad$ $\mathrm{I_{xy}} = \square$ kg m^2 $\quad$ $\mathrm{I_{xy}} = \square$ kg m^2

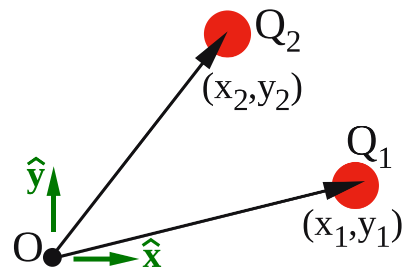
Circle the correct answer (negative, zero, or positive) for each statement about particle Q.

- When Q is in quadrant **I**, $\mathrm{I_{xy}}$ is **negative/zero/positive**.
- When Q is in quadrant **II**, $\mathrm{I_{xy}}$ is **negative/zero/positive**.
- When Q is in quadrant **III**, $\mathrm{I_{xy}}$ is **negative/zero/positive**.
- When Q is in quadrant **IV**, $\mathrm{I_{xy}}$ is **negative/zero/positive**.
- When Q is on a quadrant boundary, $\mathrm{I_{xy}}$ is **negative/zero/positive**.

13.6 ♣ Calculations: Product of inertia for a system of particles (Section 14.2).

The ***product of inertia*** of a system of particles is simply the sum of the products of inertias of each of the individual particles. For example, the product of inertia of particles Q_1 and Q_2 about point O for the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions is calculated by the formula

$$\boxed{\mathrm{I_{xy}} \;=\; \sum_{i=1}^{2} -m_i \, x_i \, y_i}$$

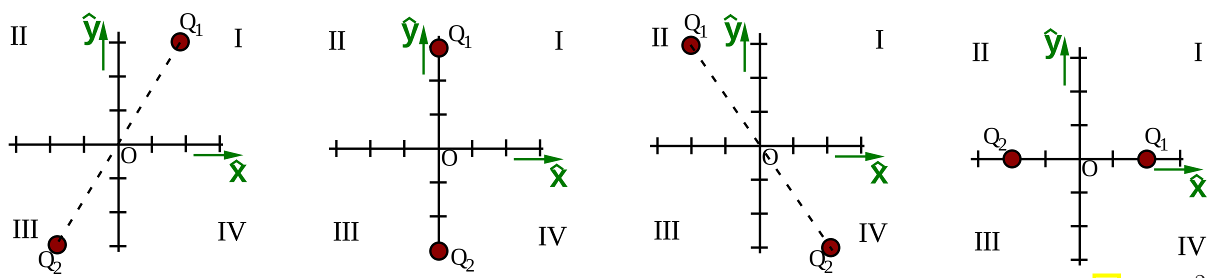


- m_i is the mass of particle Q_i $(i\!=\!1,\,2)$
- x_i is the $\hat{\mathbf{x}}$ measure of Q_i's position from O
- y_i is the $\hat{\mathbf{y}}$ measure of Q_i's position from O

For example, if $m_1 = 1$ kg, $x_1 = 3$ m, $y_1 = 1$ m, and $m_2 = 2$ kg, $x_2 = 2$ m, $y_2 = 3$ m,

$$\mathrm{I_{xy}} \;=\; -m_1\, x_1\, y_1 \;+\; -m_2\, x_2\, y_2 \;=\; -(1\text{ kg})\,(3\text{ m})\,(1\text{ m}) \;+\; -(2\text{ kg})\,(2\text{ m})\,(3\text{ m}) \;=\; -15\text{ kg m}^2$$

Knowing each particle has a mass of 1 kg and each tick-mark represents 1 m, calculate the system's ***product of inertia*** $\mathrm{I_{xy}}$ about point O for each figure below.



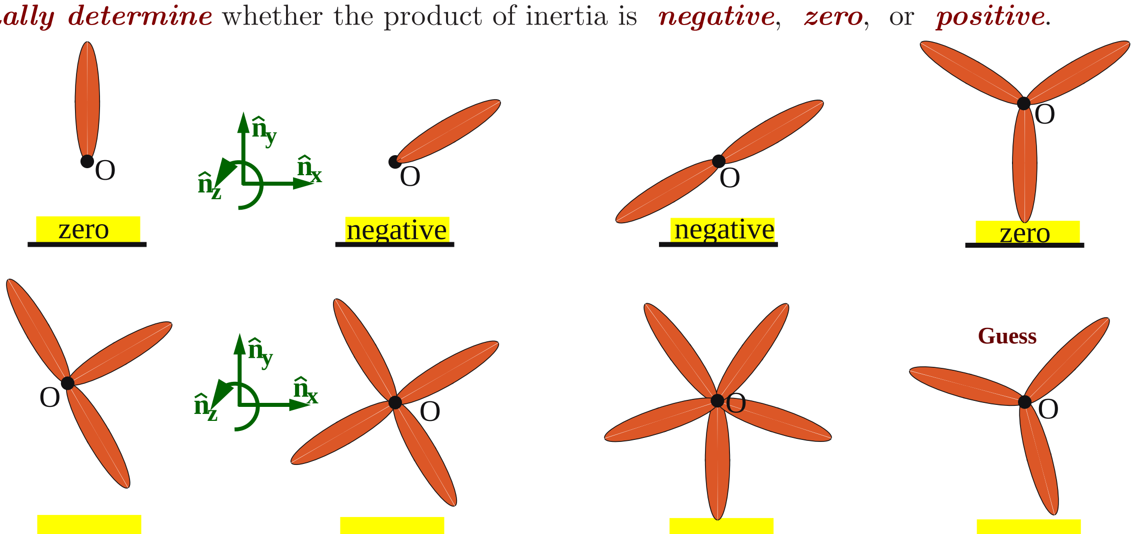
$\mathrm{I_{xy}} = \square$ kg m^2 $\quad$ $\mathrm{I_{xy}} = \square$ kg m^2 $\quad$ $\mathrm{I_{xy}} = \square$ kg m^2 $\quad$ $\mathrm{I_{xy}} = \square$ kg m^2

Circle the correct answer (negative, zero, or positive) for each of the following statements.

- When the particles are in quadrants **I** and **III**, $\mathrm{I_{xy}}$ is **negative/zero/positive**.
- When the particles are in quadrants **II** and **IV**, $\mathrm{I_{xy}}$ is **negative/zero/positive**.
- When the particles are on quadrant boundaries, $\mathrm{I_{xy}}$ is **negative/zero/positive**.

13.7 ♣ Concepts: Products of inertia of propellers (Section 14.2.2).

The following shows four uniform-density objects. For each object, consider I_{xy} the product of inertia of the object for lines that pass through point O and are parallel to $\hat{\mathbf{n}}_\mathrm{x}$ and $\hat{\mathbf{n}}_\mathrm{y}$. For each object, ***visually determine*** whether the product of inertia is ***negative***, ***zero***, or ***positive***.

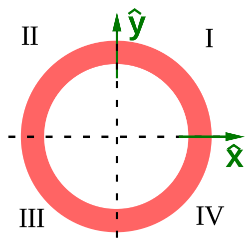


13.8 ♣ Concepts: Products of inertia – what is I_{xy}? (Section 14.2.2).

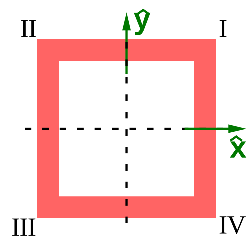
Product of inertia is a measure of the symmetry of mass distribution in two directions about a point. To investigate this concept, use your geometrical insights (not equations) to determine which of the following uniform-density objects have a negative, zero, or positive ***product of inertia*** I_{xy}.

Visually sum the mass distribution for I_{xy} in quadrants **II** and **IV** and compare that to the mass distribution in quadrants **I** and **III**. Complete each blank below with $\boxed{<}$ or $\boxed{=}$ or $\boxed{>}$.

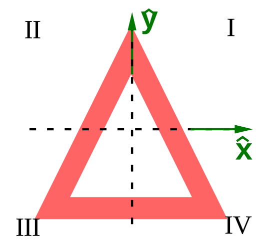
- If mass distribution in quadrants **II** + **IV** is greater than in quadrants **I** + **III**, $I_{xy} > 0$.
- If mass distribution in quadrants **II** + **IV** is smaller than in quadrants **I** + **III**, $I_{xy} < 0$.
- If mass distribution in quadrants **II** + **IV** is equal to quadrants **I** + **III**, $I_{xy} = 0$.



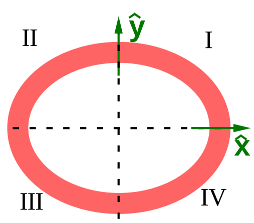
I_{xy} $\boxed{=}$ 0



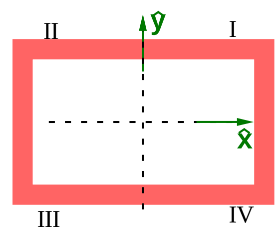
I_{xy} $\boxed{}$ 0



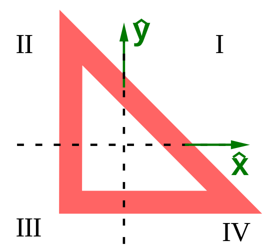
I_{xy} $\boxed{}$ 0



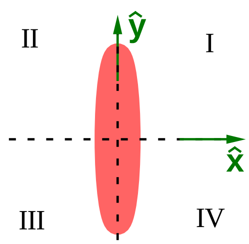
I_{xy} $\boxed{}$ 0



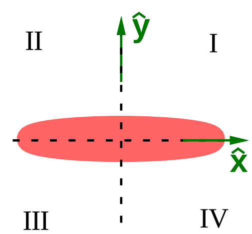
I_{xy} $\boxed{}$ 0



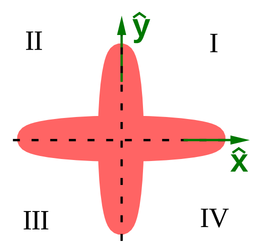
I_{xy} $\boxed{}$ 0



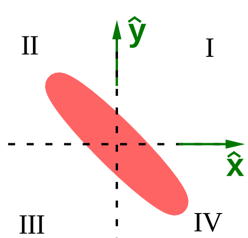
I_{xy} $\boxed{}$ 0



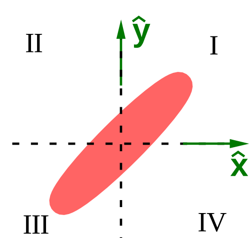
I_{xy} $\boxed{}$ 0



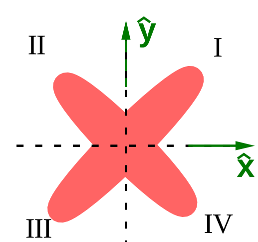
I_{xy} $\boxed{}$ 0



I_{xy} $\boxed{}$ 0



I_{xy} $\boxed{}$ 0



I_{xy} $\boxed{}$ 0

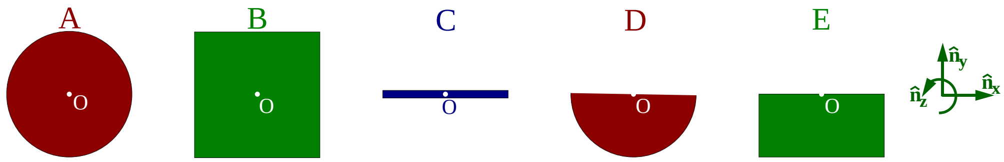
I_{xy} $\boxed{}$ 0

I_{xy} $\boxed{}$ 0

I_{xy} $\boxed{}$ 0

Purchase rattleback at www.arbor-sci.com. Explained: http://www.youtube.com/watch?v=0RyLV-Fsl4A

13.9 ♣ Conceptual understanding of moments and products of inertia (Sections 14.1.2 and 14.2.2).
Objects A, B, C, D, and E are all flat planar objects with uniform density and the ***same*** mass. The circle and semi-circle's diameter, square and rectangle's width, and thin rod's length are ***equal***.



S denotes one of A, B, C, D, or E.

36% Consider $I_{zz}^{S/O}$, S's moment of inertia about the line passing through point O and parallel to $\hat{\mathbf{n}}_z$. Knowing moment of inertia is mass $*$ distance^2, use **visual estimates** to list the objects in ascending order of $I_{zz}^{S/O}$. If two objects have the same value of $I_{zz}^{S/O}$, group them together.

Result:

Smallest						Largest

28% Consider $I_{zz}^{S/S_{cm}}$, S's moment of inertia about the line passing through S_{cm} (the mass center of S) and parallel to $\hat{\mathbf{n}}_z$. Use visual estimates to list the objects in ascending order of $I_{zz}^{S/S_{cm}}$.
Note: A and E have nearly equal $I_{zz}^{S/S_{cm}}$. The textbook's inertia appendix helps resolve their difference.

Result:

Smallest	D (**given**)					Largest

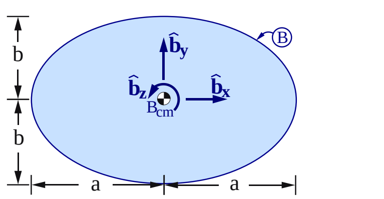
23% Consider $I_{xy}^{S/O}$, S's product of inertia for point O and unit vectors $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$. For each object, visually determine if $I_{xy}^{S/O}$ is negative ($-$), zero (0), or positive (+).

Result:

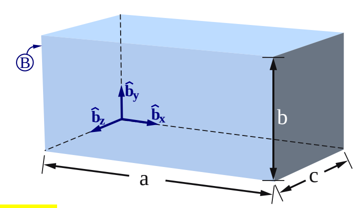
A	B	C	D	E
$-$ 0 +	$-$ 0 +	$-$ 0 +	$-$ 0 +	$-$ 0 +

13.10 ♣ Assembling inertia dyadics from the textbook's inertia appendix (Sections 16.1 and 16.5).

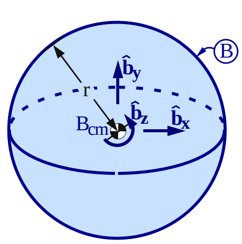
Referring to the following figures and the textbook's inertia appendix, assemble $\overset{\Rightarrow}{\mathbf{I}}^{B/B_{cm}}$ (B's inertia dyadic about its center of mass B_{cm}) for the unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$. Express results in matrix form.



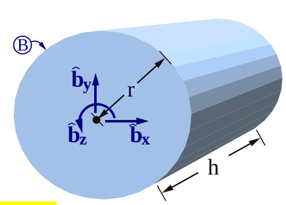
$$\begin{bmatrix} \quad & \quad & 0 \\ \quad & \boxed{\frac{1}{4}\, m\, a^2} & 0 \\ 0 & 0 & \quad \end{bmatrix}_{\hat{\mathbf{b}}_{xyz}}$$



$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & 0 \\ \quad & \quad & \frac{1}{12}\, m\, (a^2 + b^2) \end{bmatrix}_{\hat{\mathbf{b}}_{xyz}}$$



$$\quad * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\hat{\mathbf{b}}_{xyz}} = \quad \overset{\Rightarrow}{\mathbf{1}}$$



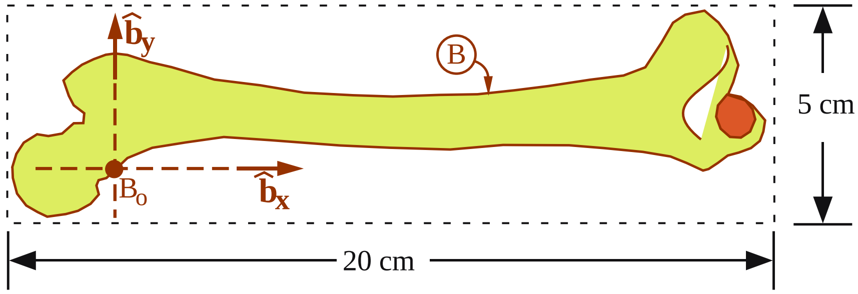
$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}_{\hat{\mathbf{b}}_{xyz}}$$

13.11 †♣ Biomechanics: Estimating mass distribution properties of a bone (Sections 14.1 and 14.2).

The following figure shows a relatively **thin**, uniform-density, rigid bone B. Right-handed orthogonal unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are fixed in B. A point B_o of B is midway though the bone ($\vec{\mathbf{r}}^{B_{cm}/B_o} \cdot \hat{\mathbf{b}}_z = 0$).

Estimate and **draw** the location of B_{cm} (B's center of mass) ◕ on the figure.

Hint: Draw horizontal (parallel to $\hat{\mathbf{b}}_x$) and vertical (parallel to $\hat{\mathbf{b}}_y$) lines passing through B_o. Similarly for B_{cm}.



Knowing the 1 kg bone fits snugly in a 20 cm by 5 cm by 1 cm box, ***visually estimate*** the following values. Note: Use each of the following values *once* in your table (so use 0 four times).

–13 –2 0 0 0 0 1 3 30 31 86 89

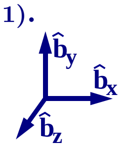
Description	Symbol	Approximate value
$\hat{\mathbf{b}}_x$ measure of B_{cm}'s position from B_o	x	7.5 cm
$\hat{\mathbf{b}}_y$ measure of B_{cm}'s position from B_o	y	1.5 cm
B's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_x$	$I_{xx}^{B/B_{cm}}$	kg cm^2
B's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_y$	$I_{yy}^{B/B_{cm}}$	kg cm^2
B's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_z$	$I_{zz}^{B/B_{cm}}$	kg cm^2
B's product of inertia about B_{cm} for $\hat{\mathbf{b}}_x$ and $\hat{\mathbf{b}}_y$	$I_{xy}^{B/B_{cm}}$	kg cm^2
B's product of inertia about B_{cm} for $\hat{\mathbf{b}}_x$ and $\hat{\mathbf{b}}_z$	$I_{xz}^{B/B_{cm}}$	kg cm^2
B's product of inertia about B_{cm} for $\hat{\mathbf{b}}_y$ and $\hat{\mathbf{b}}_z$	$I_{yz}^{B/B_{cm}}$	kg cm^2
B's moment of inertia about B_o for $\hat{\mathbf{b}}_x$	I_{xx}^{B/B_o}	kg cm^2
B's moment of inertia about B_o for $\hat{\mathbf{b}}_y$	I_{yy}^{B/B_o}	kg cm^2
B's moment of inertia about B_o for $\hat{\mathbf{b}}_z$	I_{zz}^{B/B_o}	kg cm^2
B's product of inertia about B_o for $\hat{\mathbf{b}}_x$ and $\hat{\mathbf{b}}_y$	I_{xy}^{B/B_o}	kg cm^2
B's product of inertia about B_o for $\hat{\mathbf{b}}_x$ and $\hat{\mathbf{b}}_z$	I_{xz}^{B/B_o}	kg cm^2
B's product of inertia about B_o for $\hat{\mathbf{b}}_y$ and $\hat{\mathbf{b}}_z$	I_{yz}^{B/B_o}	kg cm^2

Hint: Start with products of inertia, then smallest moments of inertia, then largest moments of inertia.
Hint: Draw an end-view of the bone to help estimate the two I_{yz}. Draw a top-view to estimate the two I_{xz}.

13.12 ♣ Dyadics and dot-products with orthogonal unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ (Section 15.1).

Non-symmetric dyadic: $\vec{\vec{\mathbf{N}}} = \hat{\mathbf{b}}_x \hat{\mathbf{b}}_x + 2\,\hat{\mathbf{b}}_y \hat{\mathbf{b}}_y + 6\,\hat{\mathbf{b}}_y \hat{\mathbf{b}}_z + 3\,\hat{\mathbf{b}}_z \hat{\mathbf{b}}_z$

Symmetric dyadic: $\vec{\vec{\mathbf{S}}} = \hat{\mathbf{b}}_x \hat{\mathbf{b}}_x + 2\,\hat{\mathbf{b}}_y \hat{\mathbf{b}}_y + 6\,\hat{\mathbf{b}}_y \hat{\mathbf{b}}_z + 6\,\hat{\mathbf{b}}_z \hat{\mathbf{b}}_y + 3\,\hat{\mathbf{b}}_z \hat{\mathbf{b}}_z$



Non-symmetric dyadic	*Symmetric dyadic*
$\vec{\vec{\mathbf{N}}} \cdot (\hat{\mathbf{b}}_x + \hat{\mathbf{b}}_y) = \boxed{\hat{\mathbf{b}}_x} + \boxed{2\,\hat{\mathbf{b}}_y}$	$\vec{\vec{\mathbf{S}}} \cdot (\hat{\mathbf{b}}_x + \hat{\mathbf{b}}_y) = \square + \square + \square$
$(\hat{\mathbf{b}}_x + \hat{\mathbf{b}}_y) \cdot \vec{\vec{\mathbf{N}}} = \square + \square + \square$	$(\hat{\mathbf{b}}_x + \hat{\mathbf{b}}_y) \cdot \vec{\vec{\mathbf{S}}} = \square + \square + \square$
$\vec{\vec{\mathbf{N}}} \cdot \text{any}\vec{\text{Vector}} = \text{any}\vec{\text{Vector}} \cdot \vec{\vec{\mathbf{N}}}$ **True/False**	$\vec{\vec{\mathbf{S}}} \cdot \text{any}\vec{\text{Vector}} = \text{any}\vec{\text{Vector}} \cdot \vec{\vec{\mathbf{S}}}$ **True/False**

Note: All inertia dyadics $\vec{\vec{\mathbf{I}}}$ are symmetric. For any vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\vec{\mathbf{I}}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\vec{\mathbf{I}}}$ and $\vec{\mathbf{u}} \cdot \vec{\vec{\mathbf{I}}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\vec{\mathbf{I}}} \cdot \vec{\mathbf{u}}$.