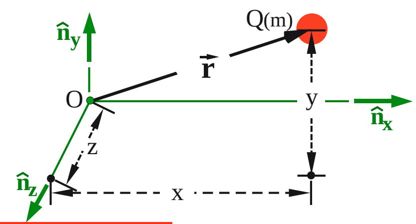
13.1 ♣ Concepts: What objects have a moment of inertia? (Section 14.1).

Consider the ***moment of inertia*** $I^{S/O}_{\hat{\mathbf{u}}\,\hat{\mathbf{u}}}$ of an object S about a point O for the unit vector $\hat{\mathbf{u}}$.
In general, for $I^{S/O}_{\hat{\mathbf{u}}\,\hat{\mathbf{u}}}$ to be a positive real number, S should be a (circle ***all*** appropriate objects):

Real number	Matrix	Set of points	Mass center of a rigid body
Vector	Point	Reference frame	Flexible body
3D orthogonal unit basis	Particle	Rigid body	System of particles and bodies

13.2 ♣ Formulas for a particle's moments and products of inertia (Sections 14.1.1 and 14.2.1).

The figure shows a particle Q of mass m and right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$.
Q's position vector from a point O is $x\,\hat{\mathbf{n}}_x + y\,\hat{\mathbf{n}}_y + z\,\hat{\mathbf{n}}_z$.



Express I_{xx} (Q's ***moment of inertia*** about O for $\hat{\mathbf{n}}_x$) in terms of some or all of m, x, y z. Similarly for I_{yy} and I_{zz}.

Express I_{xy} (Q's ***product of inertia*** about O for $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$) in terms of some or all of m, x, y z. Similarly for I_{xz} and I_{yz}.

Result:

$$I_{xx} = \square(\square^2 + \square^2) \qquad I_{yy} = \square \qquad I_{zz} = \square$$
$$I_{xy} = -\square\,\square\,\square \qquad I_{xz} = \square \qquad I_{yz} = \square$$

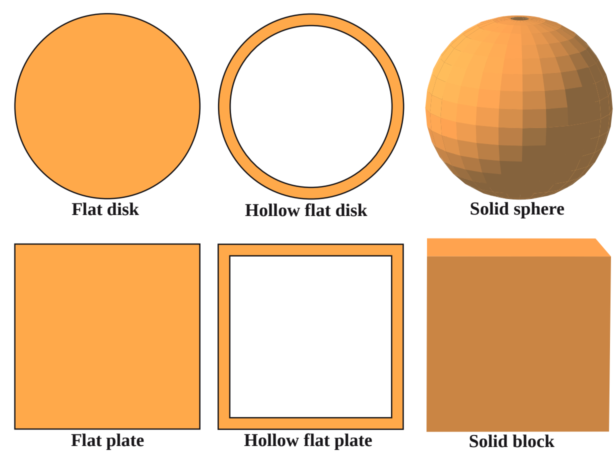
Circa 1895, Gibbs invented the ***inertia dyadic*** as a **convenient "suitcase"** for holding moments and products of inertia. Write Q's inertia dyadic about O in terms of $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and I_{ij} $(i, j = x, y, z)$.
Hint: Pattern match or refer to Section 16.1.

$$\overset{\Rightarrow}{\mathbf{I}} = I_{xx}\,\hat{\mathbf{n}}_x\,\hat{\mathbf{n}}_x + I_{xy}\,\hat{\mathbf{n}}_x\,\hat{\mathbf{n}}_y + \square\,\hat{\mathbf{n}}_x\,\hat{\mathbf{n}}_z$$
$$+ I_{xy}\,\hat{\mathbf{n}}_y\,\hat{\mathbf{n}}_x + \square\,\hat{\mathbf{n}}_y\,\hat{\mathbf{n}}_y + \square$$
$$+ \square + \square + \square$$

13.3 ♣ Moment of inertia concepts (Section 14.1.2).

Each object below has uniform density and a mass of 1 kg. One can visually determine the relative size of each object's moment of inertia I_{zz} about the line perpendicular to the plane of the paper that passes through its center of mass.

Flat disk		Hollow flat disk
Flat disk		Solid sphere
Hollow flat disk		Solid sphere
Flat plate		Hollow flat plate
Flat plate		Solid block
Hollow flat plate		Solid block
Flat disk		Flat plate
Hollow flat disk		Hollow flat plate
Solid sphere		Solid block



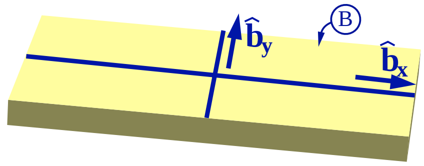
Since I_{zz} depends on mass $*$ distance^2, objects with a higher mass concentration further from the mass center have the larger moment of inertia.

13.4 ♣ Demo: How moment of inertia affects a spinning book (Section 14.1.2).

Experiment: Spin a uniform rigid body B (such as a book with a rubber-band to keep it closed) about $\widehat{\mathbf{b}}_x$, then $\widehat{\mathbf{b}}_y$, then $\widehat{\mathbf{b}}_z$. The spin can be "***neutrally stable***" (small perturbations of spin do not grow or decay – the book spins "smoothly") or "***unstable***" (small perturbations grow exponentially – the book spins crazily).

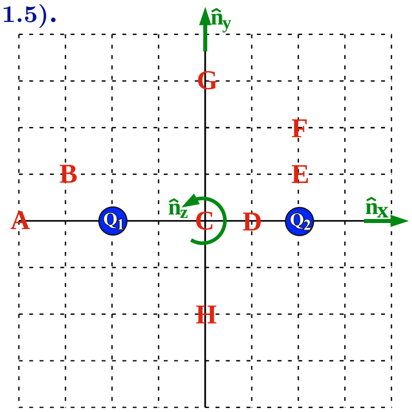
Consider I_{xx}, I_{yy}, I_{zz}, B's moments of inertia about B_{cm} (B's center of mass) for $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$, respectively.

Axis to spin book	Moment of inertia & size		Stability
$\widehat{\mathbf{b}}_x$	I_{xx}		neutrally stable
$\widehat{\mathbf{b}}_y$	I_{yy}		unstable
$\widehat{\mathbf{b}}_z$	I_{zz}		



13.5 ♣ Parallel axis theorem and moments of inertia (Section 14.1.5).

The system S shown to the right consists of particles Q_1 and Q_2, each of mass m, in a plane perpendicular to the unit vector $\widehat{\mathbf{n}}_z$.



The ***shift theorem*** (also called the ***parallel axis theorem***) shifts S's moment of inertia about S_{cm} (the ***mass center*** of S) for the unit vector $\widehat{\mathbf{n}}_z$ to an arbitrary point P in the plane using

$$I_{zz}^{S/P} \;=\; I_{zz}^{S/S_{cm}} \;+\; m^S * d^2$$

where $I_{zz}^{S/S_{cm}}$ is the system's moment of inertia about S_{cm} for $\widehat{\mathbf{n}}_z$, m^S is the mass of S, and d is the distance from S_{cm} to P.

Use the shift theorem to estimate the order of S's moment of inertia for the lines parallel to $\widehat{\mathbf{n}}_z$ that pass through points A, B, C, D, E, F, G, and H, respectively. Note: Grid lines are equally spaced.

Result:

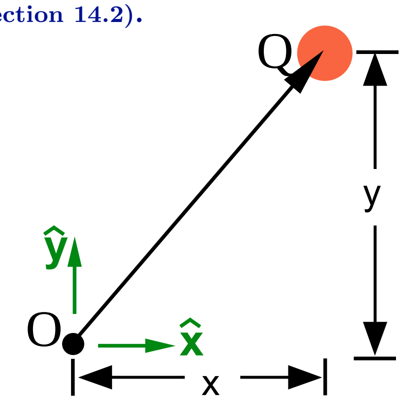
Smallest									Largest

Knowing each particle has mass $m = 1$ kg and the grid lines are spaced 1 m apart, calculate S's moment of inertia about A, B, C, D, E, F, G, and H, respectively.

Result: (in kg m^2)

$I_{zz}^{S/A}$	$I_{zz}^{S/B}$	$I_{zz}^{S/C}$	$I_{zz}^{S/D}$	$I_{zz}^{S/E}$	$I_{zz}^{S/F}$	$I_{zz}^{S/G}$	$I_{zz}^{S/H}$
		8					

13.6 ♣ Calculations: Product of inertia for a single particle (Section 14.2).



The ***product of inertia*** of a single particle Q about a point O for the $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ directions is calculated by the formula

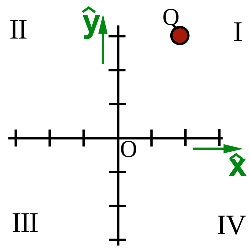
$$\boxed{\mathrm{I_{xy}} = -m\,x\,y}$$

- m is the mass of Q
- x is the $\widehat{\mathbf{x}}$ measure of Q's position from O
- y is the $\widehat{\mathbf{y}}$ measure of Q's position from O

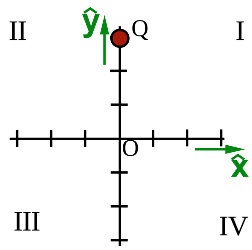
For example, if $m = 1$ kg, $x = 2$ m, and $y = 3$ m,

$$\mathrm{I_{xy}} = \text{-}(1\text{ kg})\,(2\text{ m})\,(3\text{ m}) = \text{-}6\text{ kg m}^2$$

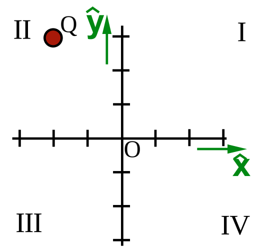
Knowing particle Q has a mass of 1 kg and each tick-mark represents 1 m, calculate Q's ***product of inertia*** $\mathrm{I_{xy}}$ about point O for each figure below.



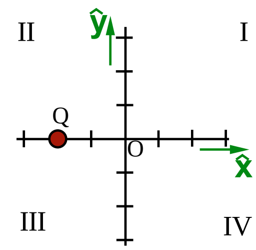
$\mathrm{I_{xy}}$ = -6 kg m^2



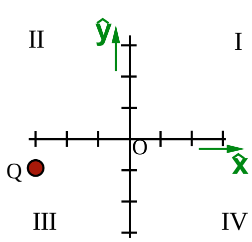
$\mathrm{I_{xy}} = \square$ kg m^2



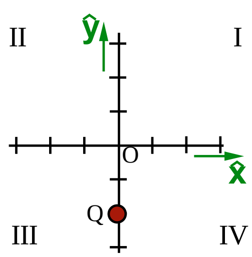
$\mathrm{I_{xy}} = \square$ kg m^2



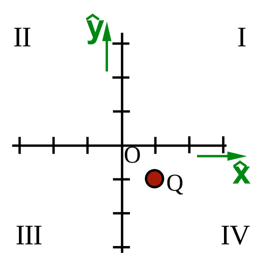
$\mathrm{I_{xy}} = \square$ kg m^2



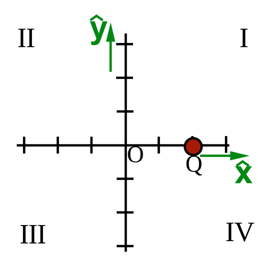
$\mathrm{I_{xy}}$ = -3 kg m^2



$\mathrm{I_{xy}} = \square$ kg m^2



$\mathrm{I_{xy}} = \square$ kg m^2



$\mathrm{I_{xy}} = \square$ kg m^2

Circle the correct answer (negative, zero, or positive) for each statement about particle Q.

- When Q is in quadrant **I**, $\mathrm{I_{xy}}$ is **negative/zero/positive**.
- When Q is in quadrant **II**, $\mathrm{I_{xy}}$ is **negative/zero/positive**.
- When Q is in quadrant **III**, $\mathrm{I_{xy}}$ is **negative/zero/positive**.
- When Q is in quadrant **IV**, $\mathrm{I_{xy}}$ is **negative/zero/positive**.
- When Q is on a quadrant boundary, $\mathrm{I_{xy}}$ is **negative/zero/positive**.

13.7 ♣ Sign conventions for products of inertia (Section 14.2.5).

There are two sign conventions ($\pm$) for products of inertia which often lead to errors. **True/False**.