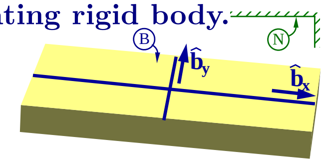
15.6 Calculate: Angular momentum and kinetic energy of a 3D rotating rigid body.



Consider a rigid body B rotating in a Newtonian reference frame N. Right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in B and parallel to B's principal inertia axes about B_{cm} (B's center of mass).

Quantity	Symbol	Type
B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_x$	I_{xx}	Constant
B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_y$	I_{yy}	Constant
B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_z$	I_{zz}	Constant
$\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ measures of ${}^N\vec{\boldsymbol{\omega}}^B$	ω_x, ω_y, ω_z	Variables

$$ {}^N\vec{\boldsymbol{\omega}}^B = \omega_x\,\widehat{\mathbf{b}}_x + \omega_y\,\widehat{\mathbf{b}}_y + \omega_z\,\widehat{\mathbf{b}}_z $$

$$ \vec{\vec{\mathrm{I}}}^{B/B_{cm}} = I_{xx}\,\widehat{\mathbf{b}}_x\widehat{\mathbf{b}}_x + I_{yy}\,\widehat{\mathbf{b}}_y\widehat{\mathbf{b}}_y + I_{zz}\,\widehat{\mathbf{b}}_z\widehat{\mathbf{b}}_z = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}_{\widehat{\mathbf{b}}_{xyz}} $$

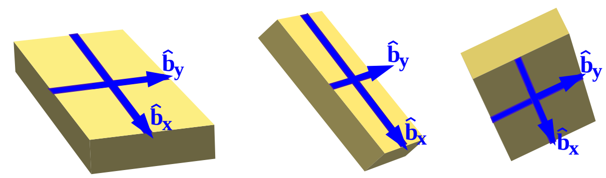
Calculate B's angular momentum about B_{cm} in N and B's rotational kinetic energy in N.

Result: $ {}^N\vec{\mathrm{H}}^{B/B_{cm}} \underset{(17.1)}{=} I_{xx}\,\omega_x\,\widehat{\mathbf{b}}_x + I_{yy}\,\omega_y\,\widehat{\mathbf{b}}_y + \boxed{}\,\widehat{\mathbf{b}}_z \qquad {}^N K^B_{\text{rotation}} \underset{(17.10)}{=} \frac{1}{2}\left(I_{xx}\,\omega_x^2 + I_{yy}\,\omega_y^2 + \boxed{}\right)$

15.7 3D spinning rigid body: Guess and check conservation of energy/angular momentum.

This problem refers to Homeworks 6.26 and 15.6.

$$ \boxed{\vec{\mathrm{M}}^{\vec{0}} \underset{(22.4)}{=} \frac{{}^N d\,\vec{\mathrm{H}}}{dt}} \Rightarrow \begin{aligned} \dot{\omega}_x &= \left[(I_{yy} - I_{zz})\,\omega_z\,\omega_y\right] / I_{xx} \\ \dot{\omega}_y &= \left[(I_{zz} - I_{xx})\,\omega_x\,\omega_z\right] / I_{yy} \\ \dot{\omega}_z &= \left[(I_{xx} - I_{yy})\,\omega_y\,\omega_x\right] / I_{zz} \end{aligned} $$



- Using your intuition and/or ***conservation of angular momentum/mechanical energy*** (Sections 22.7 and 25.2), circle the quantities below that you guess remain constant (are "conserved").
- Next, solve the ODEs in Hw 6.26 for $0 \le t \le 4$ with initial values $\omega_x = 7$, $\omega_y = 0.2$, $\omega_z = 0.2$ (use **M**otion**G**enesis, MATLAB®, or ...). Output t, ω_x, ω_y, ω_z, H_x, H_y, H_z, $H_{mag} \triangleq |\vec{\mathrm{H}}|$, and K.
- Circle the quantities that remain constant (are "***conserved***") while solving the ODEs.

ω_x	ω_y	ω_z	K
$H_x = \vec{\mathrm{H}} \cdot \widehat{\mathbf{b}}_x$	$H_y = \vec{\mathrm{H}} \cdot \widehat{\mathbf{b}}_y$	$H_z = \vec{\mathrm{H}} \cdot \widehat{\mathbf{b}}_z$	$\lvert\vec{\mathrm{H}}\rvert$

Hint: To determine if a quantity such as ω_x is constant during numerical integration, look at the numerical values of ω_x or **zoom** into the plot of ω_x vs. t and check if ω_x is constant to within a reasonable multiplier (e.g., 10) of numerical integrator accuracy. In **M**otion**G**enesis and MATLAB®, the default numerical integrator accuracy is $\approx 1\,\text{x}\,10^{-6}$.

Solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Solving 1st-order ODEs**.
Add the line: `Hx = Ixx*wx;  Hy = Iyy*wy;  Hz = Izz*wz;  Hmag = sqrt(Hx^2 + Hy^2 + Hz^2),` etc.
Add the line: `Output t sec,  Hx kg*m^2/sec,  Hy kg*m^2/sec,  Hz kg*m^2/sec,  Hmag kg*m^2/sec,` etc.
View/plot the numerical results (numbers) and determine if they stay constant to within $\approx 1\,\text{x}\,10^{-5}$.

Optional: Simulate delayed 3D spin instability for wingnut or T-handle with: $I_{xx} < I_{yy} < I_{zz}$, spin is initially mostly about $\widehat{\mathbf{b}}_y$ (intermediate moment-of-inertia axis), and $(I_{yy} - I_{xx})\,(I_{zz} - I_{yy})$ is small, e.g., by using $\omega_x(0) = 0.2\,\frac{\text{rad}}{\text{sec}}$, $\omega_y(0) = 7.0\,\frac{\text{rad}}{\text{sec}}$, $\omega_z(0) = 0.2\,\frac{\text{rad}}{\text{sec}}$, $I_{xx} = 1.9\,\text{kg}\,\text{m}^2$, $I_{yy} = 2.0\,\text{kg}\,\text{m}^2$, $I_{zz} = 3.0\,\text{kg}\,\text{m}^2$,
Video and Dzhanibekov analysis at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Spin stability**

To analytically determine if H_x is constant, time-differentiate H_x and check if the resulting expression is 0. Since $H_x = I_{xx}\,\omega_x$ and I_{xx} is constant, ω_x is constant if-and-only-if H_x is constant.

$$ \begin{aligned} \frac{d\,H_x}{dt} &= \frac{d\,(\vec{\mathrm{H}} \cdot \widehat{\mathbf{b}}_x)}{dt} \\ &\underset{(7.6)}{=} \frac{{}^N d\,\vec{\mathrm{H}}}{dt} \cdot \widehat{\mathbf{b}}_x + \vec{\mathrm{H}} \cdot \frac{{}^N d\,\widehat{\mathbf{b}}_x}{dt} \\ &= \vec{\mathrm{M}}^{\vec{0}} \cdot \widehat{\mathbf{b}}_x + \vec{\mathrm{H}} \cdot \frac{{}^N d\,\widehat{\mathbf{b}}_x}{dt} \\ &\underset{(8.1)}{=} \vec{\mathrm{H}} \cdot {}^N\vec{\boldsymbol{\omega}}^B \times \widehat{\mathbf{b}}_x \\ &= \vec{\mathrm{H}} \cdot (\omega_z\,\widehat{\mathbf{b}}_y - \omega_y\,\widehat{\mathbf{b}}_z) \\ &= (I_{yy} - I_{zz})\,\omega_y\,\omega_z \neq 0 \quad \text{since} \end{aligned} $$

$I_{yy} \neq I_{zz}$ and $\omega_y(0) \neq 0$ and $\omega_z(0) \neq 0$
[or more generally $\omega_y(t) \neq 0$ and $\omega_z(t) \neq 0$].

- Using this analytical test and a similar test for $\frac{d\,(\vec{\mathrm{H}} \cdot \vec{\mathrm{H}})}{dt}$, circle the quantities below that remain constant (are ***conserved***).

ω_x	ω_y	ω_z	
$H_x = \vec{\mathrm{H}} \cdot \widehat{\mathbf{b}}_x$	$H_y = \vec{\mathrm{H}} \cdot \widehat{\mathbf{b}}_y$	$H_z = \vec{\mathrm{H}} \cdot \widehat{\mathbf{b}}_z$	$\lvert\vec{\mathrm{H}}\rvert$
$H_1 = \vec{\mathrm{H}} \cdot \widehat{\mathbf{n}}_x$	$H_2 = \vec{\mathrm{H}} \cdot \widehat{\mathbf{n}}_y$	$H_3 = \vec{\mathrm{H}} \cdot \widehat{\mathbf{n}}_z$	

where $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are unit vectors fixed in the Newtonian frame N.