Chapter 34

Energy integrals of equations of motion

Summary

This chapter discusses ***generalized potential energy*** and its use in various ***energy integrals*** of the equations of motion, e.g., the new ***generalized dissipative energy integral***, ***generalized energy integral***, and ***generalized Hamiltonian***. Energy integrals have several purposes, including,

- An energy integral can serve as a ***check*** on the numerical accuracy of dynamic simulations.
- An energy integral can be used to ***moderate*** numerical integration to ensure global numerical integration accuracy of the energy integral to within a user-specified tolerance.

34.1 Various energy integrals of the equations of motion

When the configuration and motion of a system S in a Newtonian frame N is described by n ***generalized coordinates*** $q_1 \ldots q_n$ and n ***generalized speeds*** $u_1 \ldots u_n$, then an ***energy integral of the equations of motion*** is an equation of the form shown in eqn (1).

$$\boxed{f(q_1 \ldots q_n,\ u_1 \ldots u_n,\ t) \;=\; C} \qquad (1)$$

where C is a ***constant*** with units of energy and t is time.

This section discusses the ***generalized dissipative energy integral*** and its special cases.

Energy integral	Equation	Constant when:
Generalized dissipative energy integral	$\mathcal{E}_Z \underset{(2)}{=} \mathrm{K}_2 + \mathcal{U} + Z$	***always***
Generalized energy integral	$\mathcal{E} \underset{(5)}{=} \mathrm{K}_2 + \mathcal{U}$	$\mathcal{P}_{nonconservative} = 0,\ \sigma_R = 0$
Generalized Hamiltonian	$\mathcal{H} \underset{(6)}{=} \mathrm{K}_2 - \mathrm{K}_0 + \mathcal{U}$	$\mathcal{P}_{nonconservative} = 0,\ \sigma = 0$
Hamiltonian	$\mathrm{H} \underset{(8)}{=} \mathrm{K}_2 - \mathrm{K}_0 + \mathrm{U}$	$\mathcal{P}_{nonconservative} = 0,\ \sigma = 0,\ \mathrm{U} = \mathcal{U}$
Conservation kinetic + generalized potential	$\mathcal{C} \underset{(9)}{=} \mathrm{K} + \mathcal{U}$	$\mathcal{P}_{nonconservative} = 0,\ \sigma = 0,\ \dot{\mathrm{K}}_0 = \dot{\mathrm{K}}_1 = 0$
Conservation of mechanical energy	$\mathrm{C} \underset{(10)}{=} \mathrm{K} + \mathrm{U}$	$\mathcal{P}_{nonconservative} = 0,\ \sigma = 0,\ \dot{\mathrm{K}}_0 = \dot{\mathrm{K}}_1 = 0,\ \mathrm{U} = \mathcal{U}$

34.1.1 The generalized dissipative energy integral (Mitiguy)

The ***generalized dissipative energy integral*** states that for ***any*** system S of ν particles $Q_1 \ldots Q_\nu$ possessing p independent generalized speeds $u_1 \ldots u_p$ in a Newtonian reference frame N, one can define a ***constant*** $\mathcal{E}_Z$.

$$\boxed{\mathcal{E}_Z \;\triangleq\; \mathrm{K}_2 \,+\, \mathcal{E}_Z \,+\, Z} \qquad (2)$$

$\mathcal{E}_Z$ has units of energy.

A system is said to have p-degrees of freedom if its ***motion*** can be described by p independent variables.
The generalized dissipative energy integral is Mitiguy's improvement of an integral developed by Kane and Levinson [46, 47].

- K_2 is the kinetic energy of S in N of degree 2 in $u_1 \ldots u_p$ (see Section 34.4)
- $\mathcal{U}$ is the portion of the system that has a generalized potential energy (see Sections 34.6 and 34.7).
- Z is the energy-quantity defined by the differential equation that relates Z to σ_R [see eqn (4)] and $\mathcal{P}_{nonconservative}$ (see Section 34.6), by

$$\frac{dZ}{dt} \triangleq \sigma_R - \mathcal{P}_{nonconservative} \qquad (3)$$

The quantity σ_R in eqn (3) is defined in terms of m_i (the mass of particle Q_i) and quantities relating to $\vec{\mathbf{v}}^{Q_i}$ (Q_i's velocity in N), as

$$\sigma_R \triangleq \sum_{i=1}^{\nu} \text{m}_\text{i}\, \vec{\mathbf{v}}_\text{R}^{Q_i} \cdot \frac{{}^N\! d\vec{\mathbf{v}}_\text{t}^{Q_i}}{dt} \qquad (4)$$

σ_R for a rigid body is in eqn (29).

where $\vec{\mathbf{v}}_\text{R}^{Q_i}$ is the portion of $\vec{\mathbf{v}}^{Q_i}$ that contains the independent generalized speeds $u_1 \ldots u_p$; $\vec{\mathbf{v}}_\text{t}^{Q_i}$ is the portion of $\vec{\mathbf{v}}^{Q_i}$ that does ***not*** contain $u_1 \ldots u_p$; and $\frac{{}^N\! d\, \vec{\mathbf{v}}_\text{t}^{Q_i}}{dt}$ is the time-derivative in N of $\vec{\mathbf{v}}_\text{t}^{Q_i}$.

34.1.2 The generalized energy integral

When $\sigma_R = 0$ and $\mathcal{P}_{nonconservative} = 0$, $Z = 0$ satisfies eqn (3), so eqn (2) simplifies to eqn (5). An example of conservation of this ***generalized energy integral*** is in Section 34.2.

$$\boxed{\mathcal{E} = \text{K}_2 + \mathcal{U}} \qquad (5)$$

34.1.3 The Hamiltonian integrals

For a system S having p independent generalized speeds $u_1 \ldots u_p$ in a Newtonian frame N, the ***generalized Hamiltonian*** $\mathcal{H}$ defined in eqn (6) is conserved (constant) if both $\mathcal{P}_{nonconservative}$ (see Section 34.6) and the quantity σ (defined right) are zero.

$$\boxed{\mathcal{H} \triangleq \text{K}_2 - \text{K}_0 + \mathcal{U}} \qquad (6)$$

$$\sigma \triangleq \sum_{i=1}^{\nu} \text{m}_\text{i}\, \vec{\mathbf{v}}^{Q_i} \cdot \frac{{}^N\! d\, \vec{\mathbf{v}}_\text{t}^{Q_i}}{dt} \qquad (7)$$

σ for a rigid body is in eqn (30).

K_2 and $\mathcal{U}$ have the same meaning as in Section 34.1.1 and K_0 is the kinetic energy of S in N of degree 0 in $u_1 \ldots u_p$ (see Section 34.4). Note: The quantity σ defined in terms of $\vec{\mathbf{v}}^{Q_i}$ is **slightly different** than σ_R which is defined in terms of $\vec{\mathbf{v}}_\text{R}^{Q_i}$.

When S also possesses a potential energy U, (i.e., the generalized potential energy $\mathcal{U}$ of Section 34.6 is **equal** to the potential energy U of Section 34.7), a **conserved** quantity called the ***Hamiltonian*** is defined as shown in eqn (8).

$$\boxed{\text{H} \triangleq \text{K}_2 - \text{K}_0 + \text{U}} \qquad (8)$$

34.1.4 Conservation of kinetic and generalized potential energy

A system S is said to conserve kinetic and generalized potential energy when, in addition to satisfying the conditions in eqn (6), $\dot{\text{K}}_0 = \dot{\text{K}}_1 = 0$ (or equivalently, $\dot{\text{K}} = \dot{\text{K}}_2$). Under these conditions, the sum of kinetic energy and generalized potential energy of S in N is equal to a constant $\mathcal{C}$, as shown in eqn (9).

$$\boxed{\mathcal{C} = \text{K} + \mathcal{U}} \qquad (9)$$

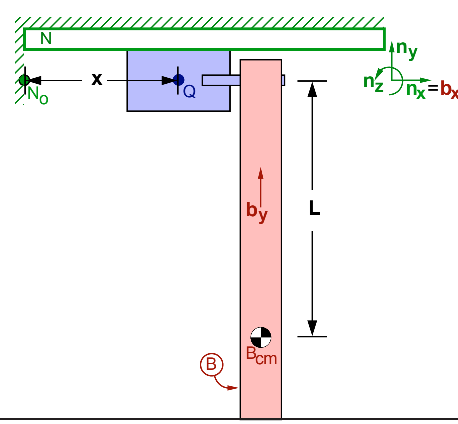
34.1.5 Conservation of mechanical energy

A system S is said to ***conserve mechanical energy*** when, in addition to satisfying the conditions in equation (9), S possesses a potential energy U, (i.e., the generalized potential energy $\mathcal{U}$ of Section 34.6 is **equal** to the potential energy U of Section 34.7). Under these conditions the sum of kinetic energy and potential energy of S in N is equal to a constant C, as shown in eqn (10).

$$\boxed{\text{C} = \text{K} + \text{U}} \qquad (10)$$

34.2 Example: Conservation of generalized energy

The figure to the right shows a rigid body B attached by a revolute joint to a hoist Q which slides on a horizontal track that is fixed in a Newtonian reference frame N. The distance of Q from a point N_o fixed in N is controlled by a translational motor so that x is a ***specified*** (i.e., ***prescribed*** or ***known***) function of time. The rotational motion of B about $\widehat{\mathbf{n}}_\text{x}$ is the system's one-degree of freedom.



Right-handed orthogonal unit vectors $\widehat{\mathbf{n}}_\text{x}$, $\widehat{\mathbf{n}}_\text{y}$, $\widehat{\mathbf{n}}_\text{z}$ are fixed in N with $\widehat{\mathbf{n}}_\text{x}$ horizontally-right and $\widehat{\mathbf{n}}_\text{y}$ vertically-upward. Right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_\text{x}$, $\widehat{\mathbf{b}}_\text{y}$, $\widehat{\mathbf{b}}_\text{z}$ are fixed in B. Initially, $\widehat{\mathbf{b}}_\text{i} = \widehat{\mathbf{n}}_\text{i}$ ($i = \text{x}, \text{y}, \text{z}$) and then B is subjected to a right-hand rotation about $\widehat{\mathbf{b}}_\text{x} = \widehat{\mathbf{n}}_\text{x}$ by an amount θ.

Quantity	Symbol	Type	Value
Mass of Q	m^Q	Constant	1.0 kg
Mass of B	m^B	Constant	2.0 kg
Distance from revolute joint to B_{cm} (B's mass center)	L	Constant	0.5 m
B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_x$	I	Constant	$0.041\bar{6}$ kg∗m^2
Earth's gravitational constant	g	Constant	9.8 m/s^2
Distance between Q and N_o (a point fixed in N)	$x(t)$	**Specified**	$3* \cos(t)$
Angle from $\widehat{\mathbf{n}}_y$ to $\widehat{\mathbf{b}}_y$ with $^+\widehat{\mathbf{n}}_x$ sense	$\theta(t)$	Variable	30° (initial)

The equation governing B's rotational motion in N is[1]

$$\ddot{\theta} \;+\; \frac{\mathrm{m}^B\, g\, L}{I \,+\, \mathrm{m}^B\, L^2}\, \sin(\theta) \;=\; 0$$

This system's ***generalized potential energy*** is B's gravitational potential energy[2]

$$\mathcal{U} \;=\; \text{-}\mathrm{m}^B\, g\, L \cos(\theta)$$

The system's kinetic energy and kinetic energies of degree 0 and 2 are

$$\mathrm{K} \;=\; \frac{1}{2}\,(\mathrm{m}^B + \mathrm{m}^Q)\,\dot{x}^2 \;+\; \frac{1}{2}\,(I + \mathrm{m}^B\, L^2)\,\dot{\theta}^2 \qquad \mathrm{K}_0 \;=\; \frac{1}{2}\,(\mathrm{m}^B + \mathrm{m}^Q)\,\dot{x}^2 \qquad \mathrm{K}_2 \;=\; \frac{1}{2}\,(I + \mathrm{m}^B\, L^2)\,\dot{\theta}^2$$

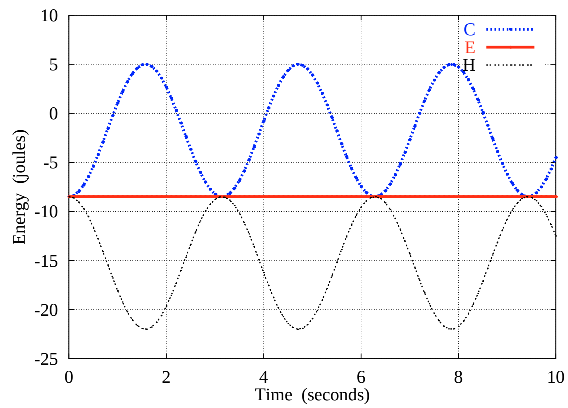
Expressions for $\mathcal{C}$ (the sum of K and $\mathcal{U}$), the generalized energy $\mathcal{E}$, and the generalized Hamiltonian $\mathcal{H}$, are[a]

$$\mathcal{C} \;\underset{(9)}{=}\; \mathrm{K} \;+\; \mathcal{U}$$

$$\mathcal{E} \;\underset{(5)}{=}\; \mathrm{K}_2 \;+\; \mathcal{U}$$

$$\mathcal{H} \;\underset{(6)}{=}\; \mathrm{K}_2 \;-\; \mathrm{K}_0 \;+\; \mathcal{U}$$

By simulating the motion of the system for 10 seconds and plotting $\mathcal{C}$, $\mathcal{E}$, and $\mathcal{H}$, one sees that $\mathcal{E}$ is **constant** whereas $\mathcal{C}$ and $\mathcal{H}$ are **not constant**.



[a]This system does ***not*** possess a potential energy. Hence, it cannot conserve mechanical energy or Hamiltonian.

The point of this simple example is to demonstrate that for this system, generalized energy $\mathcal{E}$ is an integral of the equations of motion whereas the generalized Hamiltonian $\mathcal{H}$ and the sum $\mathrm{K} + \mathcal{U}$ are not. The reason that the generalized energy is constant follows directly from the fact that both σ_R and $\mathcal{P}_{nonconservative}$ are zero, hence equation (2) simplifies to equation (5). The reason that the generalized Hamiltonian $\mathcal{H}$ varies follows directly from the fact that (see Section 34.1.3)

$$\sigma \;=\; (\mathrm{m}^B + \mathrm{m}^Q)\,\dot{x}\,\ddot{x}$$

which means that $\mathcal{H}$ is not constant unless $\ddot{x} = 0$.

If it were the case that $\ddot{x} = 0$ ($\dot{x} = $ constant), $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{C}$ would only differ by a constant because with $\dot{x}$ constant, K_0 is also constant. In addition, with $\ddot{x} = 0$, the force that causes Q to translate, [equal to $(\mathrm{m}^Q + \mathrm{m}^B)\,\ddot{x}$], would be zero so that potential energy would exist and be equal to generalized potential energy, i.e., $\mathrm{U} \;=\; \mathcal{U}$. In light of Sections 34.1.3 and 25.2, this would also mean that the Hamiltonian and mechanical energy would be constant.

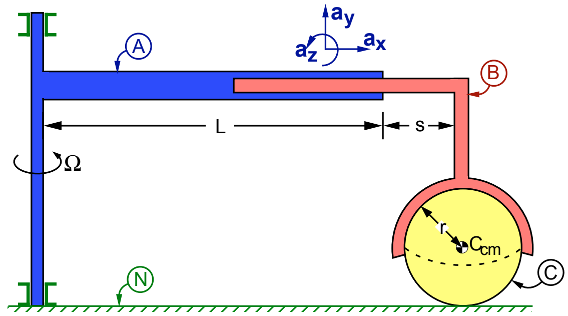
[1]One advantage of forming equations of motion with "system methods" (e.g., Lagrange & Kane) as opposed to free-body methods is that system methods can eliminate from its equations the force that causes Q to translate with a specified $x(t)$.

[2]This system does ***not*** have a potential energy because the force that causes Q to translate is time-dependent.

34.3 Example: Energy integrals of equations of motion

The robotic positioning device to the right consists of three rigid bodies A, B, and C. Body C is a uniform solid sphere that ***rolls*** on a flat Earth-fixed horizontal plane N.

Body B is relatively light and consists of two rigidly connected parts: a hemispherical housing that connects it to C; and a long extensionally-***flexible*** tube that allows translation (but not rotation) of B relative to A. Body A is made to rotate by a motor at a ***specified*** rate about an Earth-fixed vertical shaft. Right-handed orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ are fixed in A with $\widehat{\mathbf{a}}_x$ parallel to the tube and $\widehat{\mathbf{a}}_y$ vertically-upward.



Quantity	Symbol	Type	Values(s)
Mass of C	m	Constant	1 kg
Radius of C	r	Constant	0.1 m
Distance from vertical axis to distal end of A	L	Constant	1 m
Linear spring constant modeling flexibility in tube	k	Constant	$200\ \frac{\text{N}}{\text{m}}$ or ∞
Linear viscous damping constant for fluid between B and C	b	Constant	0 or $2\ \frac{\text{N*s}}{\text{m}}$
Known rate of rotation of A in N about vertical axis	$\Omega(t)$	**Specified**	$4\ \frac{\text{rad}}{\text{sec}}$ or $0.5 * t\ \frac{\text{rad}}{\text{sec}}$
Stretch of spring that models flexibility in tube	$s(t)$	Variable	$0\ \frac{\text{m}}{\text{s}}$ (initial)
$\widehat{\mathbf{a}}_x$ measure of C's angular velocity in N	$\omega_x(t)$	Variable	varies (constrained)
$\widehat{\mathbf{a}}_y$ measure of C's angular velocity in N	$\omega_y(t)$	Variable	$0\ \frac{\text{rad}}{\text{sec}}$ (initial)
$\widehat{\mathbf{a}}_z$ measure of C's angular velocity in N	$\omega_z(t)$	Variable	varies (constrained)

Equations of motion for this rolling system are[3]

$$\omega_x = \text{-}\frac{L+s}{r}\,\Omega \qquad \text{Rolling constraint equation}$$

$$\omega_z = \text{-}\frac{1}{r}\,\dot{s} \qquad \text{Rolling constraint equation}$$

$$0.4\,\text{m}\,r^2\,\dot{\omega}_y \;+\; b\,\omega_y = b\,\Omega \qquad \text{Kane's equation for generalized speed } \omega_y$$

$$1.4\,\text{m}\,r^2\,\ddot{s} \;+\; b\,\dot{s} \;+\; \left(k - 1.4\text{m}\Omega^2\right) r^2\,s = 1.4\,\text{m}\,r^2\,L\,\Omega^2 \qquad \text{Kane's equation for generalized speed } \dot{s}$$

This system's ***generalized potential energy*** is the spring's potential energy[a]

$$\mathcal{U} = \frac{1}{2}\,k\,s^2$$

[a]Even with no damping ($b = 0$), this system does ***not*** have a potential energy because the torque that causes A to rotate is time-dependent.

The system's kinetic energy and kinetic energies of degree 0 and 2 are

$$\text{K} = 0.2\,\text{m}\left[3.5\,(L+s)^2\,\Omega^2 + 3.5\,\dot{s}^2 + r^2\,\omega_y^2\right] \qquad \text{K}_0 = 0.7\,\text{m}\,(L+s)^2\,\Omega^2 \qquad \text{K}_2 = 0.2\,\text{m}\,(3.5\,\dot{s}^2 + r^2\,\omega_y^2)$$

By simulating this system's motion for 4 seconds for various values of Ω, k, and b and plotting energy quantities as shown in Figure 34.1, one can make the following observations:

- The ***generalized dissipative energy integral*** $\mathcal{E}_Z = \text{K}_2 + \mathcal{U} + Z$ is constant for ***all*** simulations.
- The ***generalized energy integral*** $\mathcal{E} = \text{K}_2 + \mathcal{U}$ is constant when there is no damping and $\dot{s} = 0$ (the tube is rigid). Section 34.1.1 describes the conditions in which $\mathcal{E}$ is constant.
  Since $\sigma_R = \text{-}1.4\,\text{m}\,\Omega^2\,(L+s)\,\dot{s}$, and $\mathcal{P}_{nonconservative}$ is zero when $b = 0$, $\mathcal{E}$ is constant when $\dot{s} = 0$ and $b = 0$.
- The ***generalized Hamiltonian*** $\mathcal{H} = \text{K}_2 - \text{K}_0 + \mathcal{U}$ is constant when there is no damping and Ω is constant. Section 34.1.3 describes the conditions in which $\mathcal{H}$ is constant.
  Since $\sigma = 1.4\,\text{m}\,\Omega\,\dot{\Omega}\,(L+s)^2$, and $\mathcal{P}_{nonconservative}$ is zero when $b = 0$, $\mathcal{H}$ is constant when $\dot{\Omega} = 0$ and $b = 0$.
- The sum of kinetic and generalized potential energy $\mathcal{C} = \text{K} + \mathcal{U}$ is ***not constant*** for any of the simulations.

[3]It is advantageous to form equations of motion with Kane's method because Kane's method eliminates from its equation the torque that causes A to rotate at the specified rate $\Omega(t)$ ***and*** the forces (or Lagrange multipliers) that cause C to roll.

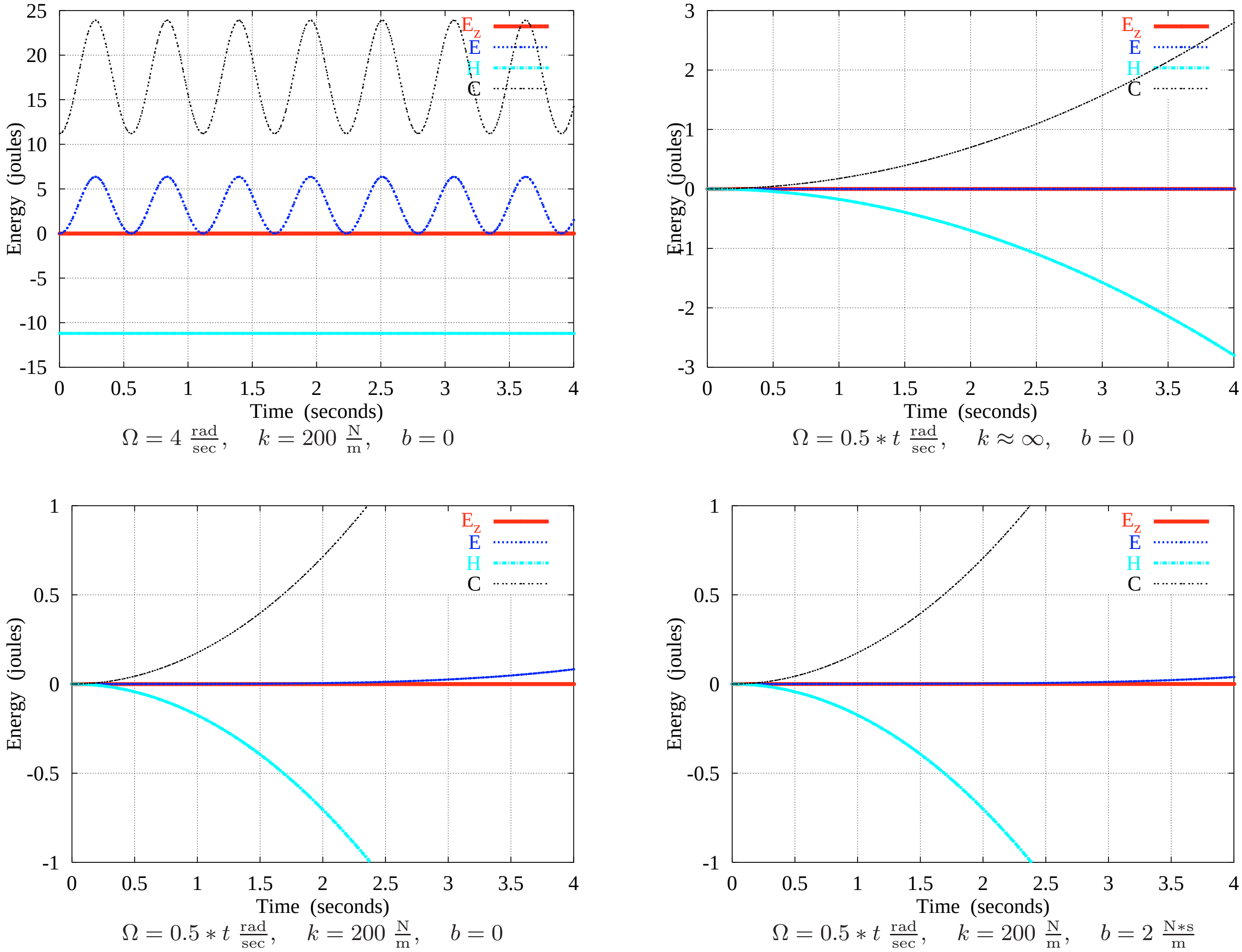
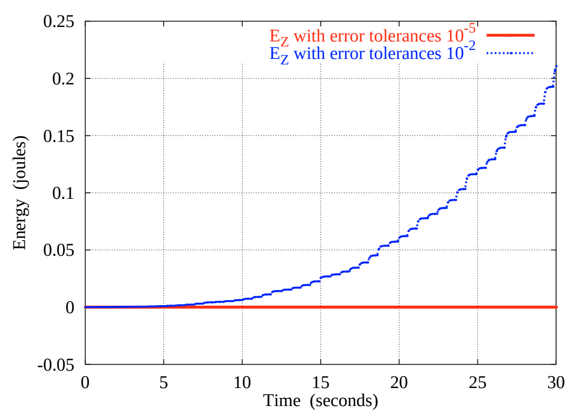


Figure 34.1: Time histories of $\mathcal{E}_Z$, $\mathcal{E}$, $\mathcal{H}$, and $\mathcal{C}$

Checking numerical integration accuracy

An energy integral can check the numerical accuracy of dynamic simulations. To demonstrate this, two simulations were run for 30 sec. Both simulations used $\Omega = 0.5 * t \, \frac{\text{rad}}{\text{sec}}$, $k = 200 \, \frac{\text{N}}{\text{m}}$, $b = 2 \, \frac{\text{N*s}}{\text{m}}$, and a numerical integration step of 0.1 sec. The numerical integration error tolerances on the first simulation were 1.0×10^{-5} whereas the second used 1.0×10^{-2}. Since $\mathcal{E}_Z$ is constant for ***any*** system governed by $\vec{\mathbf{F}} = \text{m}\,\vec{\mathbf{a}}$, one can see that the simulation results for the second simulation were less accurate than the first.



34.4 Kinetic energy terms

The kinetic energy K of a system S possessing p independent generalized speeds $u_1 \dots u_p$ in a reference frame N can be expressed in terms of portions of K that are of degree 0, 1, 2 in $u_1 \dots u_p$ as shown right.

$$\boxed{\mathrm{K} \;=\; \mathrm{K}_0 \;+\; \mathrm{K}_1 \;+\; \mathrm{K}_2} \qquad (11)$$

To precisely define K_0, K_1, K_2, consider a particle Q_i moving in frame N with a velocity $\vec{\mathbf{v}}^{Q_i}$. As shown in eqn (28.10), $\vec{\mathbf{v}}^{Q_i}$ can always be written in terms of $\frac{\partial\, {}^N\vec{\mathbf{v}}^{P_i}}{\partial u_r}$ (the u_r partial velocity of Q_i in N) and $\vec{\mathbf{v}}_{\mathrm{t}}^{Q_i}$ (the portion of $\vec{\mathbf{v}}^{Q_i}$ that does *not* contain $u_1 \dots u_p$) as shown in eqn (12), where $\vec{\mathbf{v}}_{\mathrm{R}}^{Q_i}$ is defined as the portion of $\vec{\mathbf{v}}^{Q_i}$ that contains $u_1 \dots u_p$.

$$\vec{\mathbf{v}}^{Q_i} \;=\; \underbrace{\sum_{r=1}^{p} \frac{\partial\, \vec{\mathbf{v}}^{Q_i}}{\partial u_r}\, u_r}_{\vec{\mathbf{v}}_{\mathrm{R}}^{Q_i}} \;+\; \vec{\mathbf{v}}_{\mathrm{t}}^{P_i} \qquad (12)$$

When a system S consists of ν particles Q_i $(i = 1 \dots \nu)$ of mass $\mathrm{m_i}$, K_0, K_1, K_2 are defined below.[4] For a rigid body B, more useful expressions for K_0, K_1, K_2 are also provided.

$$\begin{aligned}
\mathrm{K}_0 &\triangleq \tfrac{1}{2} \sum_{i=1}^{\nu} \mathrm{m_i}\, \vec{\mathbf{v}}_{\mathrm{t}}^{Q_i} \cdot \vec{\mathbf{v}}_{\mathrm{t}}^{Q_i} & \mathrm{K}_0 &= \tfrac{1}{2}\, \mathrm{m}^B\, \vec{\mathbf{v}}_{\mathrm{t}}^{B_p} \cdot \vec{\mathbf{v}}_{\mathrm{t}}^{B_p} \;+\; \tfrac{1}{2}\, \vec{\boldsymbol{\omega}}_{\mathrm{t}} \cdot \vec{\vec{\mathbf{I}}}^{B/B_p} \cdot \vec{\boldsymbol{\omega}}_{\mathrm{t}} \\
\mathrm{K}_1 &\triangleq \sum_{i=1}^{\nu} \mathrm{m_i}\, \vec{\mathbf{v}}_{\mathrm{t}}^{Q_i} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} & \mathrm{K}_1 &= \mathrm{m}^B\, \vec{\mathbf{v}}_{\mathrm{t}}^{B_p} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{B_p} \;+\; \vec{\boldsymbol{\omega}}_{\mathrm{t}} \cdot \vec{\vec{\mathbf{I}}}^{B/B_p} \cdot \vec{\boldsymbol{\omega}}_{\mathrm{R}} \qquad (13) \\
\mathrm{K}_2 &\triangleq \tfrac{1}{2} \sum_{i=1}^{\nu} \mathrm{m_i}\, \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} & \mathrm{K}_2 &= \tfrac{1}{2}\, \mathrm{m}^B\, \vec{\mathbf{v}}_{\mathrm{R}}^{B_p} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{B_p} \;+\; \tfrac{1}{2}\, \vec{\boldsymbol{\omega}}_{\mathrm{R}} \cdot \vec{\vec{\mathbf{I}}}^{B/B_p} \cdot \vec{\boldsymbol{\omega}}_{\mathrm{R}}
\end{aligned}$$

- m^B is the mass of B and $\vec{\vec{\mathbf{I}}}^{B/B_p}$ is B's inertia dyadic about B_p.
- B_p is either the mass center of B *or* a point fixed in both B and N. $\vec{\mathbf{v}}^{B_p}$ is B_p's velocity in N.
- $\vec{\mathbf{v}}_{\mathrm{R}}^{B_p}$ is the portion of $\vec{\mathbf{v}}^{B_p}$ that contains $u_1 \dots u_p$ whereas $\vec{\mathbf{v}}_{\mathrm{t}}^{B_p}$ is the portion ***without*** $u_1 \dots u_p$.
- $\vec{\boldsymbol{\omega}}_{\mathrm{R}}$ is the portion of $\vec{\boldsymbol{\omega}}$ (B's angular velocity in N) that contains $u_1 \dots u_p$.
- $\vec{\boldsymbol{\omega}}_{\mathrm{t}}$ is the portion of $\vec{\boldsymbol{\omega}}$ ***without*** $u_1 \dots u_p$.

34.5 Generalized power and generalized work

The ***generalized power*** of the resultant of all forces $\vec{\mathbf{F}}^Q$ on a point Q in a reference frame N is denoted $\mathcal{P}^Q$ and is defined in terms of $\vec{\mathbf{v}}_{\mathrm{R}}^Q$ as

$$\mathcal{P}^Q \;\triangleq\; \vec{\mathbf{F}}^Q \cdot \vec{\mathbf{v}}_{\mathrm{R}}^Q \qquad (14)$$

The generalized power of a set S of forces $\vec{\mathbf{F}}^{Q_1} \dots \vec{\mathbf{F}}^{Q_\nu}$ that act on points $Q_1 \dots Q_\nu$, respectively, is denoted $\mathcal{P}$ and is defined as[a]

$$\mathcal{P} \;\triangleq\; \sum_{i=1}^{\nu} \mathcal{P}^{Q_i} \qquad (15)$$

[a] An alternate expression is $\mathcal{P} = \sum_{r=1}^{p} \mathcal{F}_r * u_r$ where $\mathcal{F}_r$ $(r = u_1, \dots, u_p)$ are Kane's generalized forces [36, p. 99]. This derivation is found in equation (22).

The ***generalized work*** of a set S of forces in reference frame N is denoted $\mathcal{W}$ and is defined by an integral (or differential equation) that relates it to $\mathcal{P}$ as

$$\boxed{\mathcal{W} \;\triangleq\; \int \mathcal{P}\, dt \quad \text{or} \quad \frac{d\mathcal{W}}{dt} \;\triangleq\; \mathcal{P}} \qquad (16)$$

34.6 Generalized potential energy

In certain situations, the integral in equation (16) results in an expression that is a function of ***only*** configuration (position and orientation), i.e., it is not a function of motion or an explicit function of time. When this occurs, the ***negative*** of the integral is called a ***generalized potential energy*** of S in N, i.e.,

$$\mathcal{U} \;=\; -\mathcal{W} \quad \text{if and only if } \mathcal{W} \text{ is solely a function of configuration} \qquad (17)$$

[4] These definitions are equivalent, but computationally more efficient, to those found in [36, pg. 151].

As mentioned in Section 34.5, S is a set of forces and $\mathcal{P}$ is calculated with a sum. It is helpful to split $\mathcal{P}$ into two terms called the "conservative generalized power" and the "nonconservative generalized power" so that generalized work can be written as shown in equation (18).

$$\begin{aligned} \mathcal{P} &= \mathcal{P}_{conservative} + \mathcal{P}_{conservative} \\ \mathcal{W} &\underset{(16)}{=} \int \mathcal{P}_{conservative} * dt + \int \mathcal{P}_{nonconservative} * dt \end{aligned} \tag{18}$$

By definition, $\mathcal{P}_{conservative}$ is the sum of terms in $\mathcal{P}$ whose time-integral results in an expression that is a function of ***only*** configuration, and $\mathcal{P}_{nonconservative}$ is the remaining terms in $\mathcal{P}$. When $\mathcal{P}_{conservative}$ is non-zero, the portion of the system that has a generalized potential energy is denoted $\mathcal{U}$ and is defined as

$$\mathcal{U} \triangleq -\int \mathcal{P}_{conservative} * dt \tag{19}$$

In view of equations (18) and (19), the generalized work of S in N can be expressed

$$\mathcal{W} \underset{(18,19)}{=} -\mathcal{U} + \int \mathcal{P}_{nonconservative} * dt \tag{20}$$

34.7 *Classical* vs. *generalized* power, work, and potential energy

The following table compares the classical definitions of power, work, and potential energy to their generalized counterparts. The small differences in these definitions are important in forming integrals of the equation of motion.

Quantity	Classical definition	Generalized definition
Power	$\text{P} \triangleq \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$	$\mathcal{P} \triangleq \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}_\text{R}$
Work	$\text{W} \triangleq \int \text{P} * dt$	$\mathcal{W} \triangleq \int \mathcal{P} * dt$
Potential energy	$\text{U} \triangleq -\text{W}$ (if and only if W is solely a function of configuration)	$\mathcal{U} \triangleq -\mathcal{W}$ (if and only if $\mathcal{W}$ is solely a function of configuration)

As is apparent, the classical definition of power uses $\vec{\mathbf{v}}$ whereas the definition of generalized power uses $\vec{\mathbf{v}}_\text{R}$. In view of equation (12), the difference between these two definitions is $\vec{\mathbf{F}} \cdot \vec{\mathbf{v}}_\text{t}$ where $\vec{\mathbf{v}}_\text{t}$ is the portion of $\vec{\mathbf{v}}$ that does *not* contain $u_1 \dots u_p$. As a result, the differences between the classical and generalized definitions of power, work, and potential energy are associated with actuators or motors that move parts of a system at a ***specified*** (i.e, ***prescribed*** or ***known***) rate. Consequently, there are situations where $\mathcal{U}$ exists but U does not (see Section 34.2).

When $\vec{\mathbf{v}}_\text{t}$ does not contribute to the power of gravitation, electrostatics, or elastic forces, the generalized potential $\mathcal{U}$ and potential energy U are identical.

34.8 Generalized forces and potential energy

Consider a system S having n generalized coordinates $q_1 \dots q_n$, n generalized speeds $u_1 \dots u_n$, and m motion constraints so there are $p \triangleq n - m$ independent generalized speeds $u_1 \dots u_p$ in a Newtonian reference frame. When a generalized potential energy $\mathcal{U}$ exists, the generalized forces $\mathcal{F}_r$ are related to $\mathcal{U}$ by

$$\mathcal{F}_r = -\sum_{s=1}^{n} \frac{\partial \mathcal{U}}{\partial q_s} * \frac{\partial \dot{q}_s}{\partial u_r} \quad (r = u_1, \dots, u_p) \qquad \text{If } m = 0 \text{ (no constraints) and } u_r \triangleq \dot{q}_r, \text{ then: } \quad \mathcal{F}_r = \frac{-\partial \mathcal{U}}{\partial q} \quad (r = 1, \dots, n) \tag{21}$$

If a potential energy U exists, equation (21) can be modified by replacing $\mathcal{U}$ with U. Equation (21) can be used to calculate the unconstrained (holonomic) generalized forces by considering $\dot{q}_s$ as a function of $u_1 \ldots u_n$ instead of $u_1 \ldots u_p$.

Optional: Proof of generalized forces from potential energy

To establish the validity of equation (21), start by noting that the generalized power $\mathcal{P}$ of a set S of forces $\vec{\mathbf{F}}^{Q_1} \ldots \vec{\mathbf{F}}^{Q_\nu}$ that act on points $Q_1 \ldots Q_\nu$, respectively, is equal to

$$\mathcal{P} \underset{(14\,15)}{=} \sum_{i=1}^{\nu} \vec{\mathbf{F}}^{Q_i} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \underset{(12)}{=} \sum_{i=1}^{\nu} \vec{\mathbf{F}}^{Q_i} \cdot \left(\sum_{r=1}^{p} \frac{\partial \vec{\mathbf{v}}^{Q_i}}{\partial u_r} * u_r \right) = \sum_{r=1}^{p} \left(\sum_{i=1}^{\nu} \vec{\mathbf{F}}^{Q_i} \cdot \frac{\partial \vec{\mathbf{v}}^{Q_i}}{\partial u_r} \right) * u_r = \sum_{r=1}^{p} \mathcal{F}_r * u_r \qquad (22)$$

where $\mathcal{F}_r$ are Kane's generalized forces [36, p. 99]. In view of equations (16) and (17), $\mathcal{P}$ and $\mathcal{U}$ are related, and with equation (22), the generalized potential energy and generalized forces are related by

$$\frac{d\mathcal{U}}{dt} \underset{(16\,17)}{=} -\mathcal{P} \underset{(22)}{=} -\begin{bmatrix} \mathcal{F}_1 & .. & \mathcal{F}_p \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \qquad (23)$$

By definition, $\mathcal{U}$ is solely a function of configuration, so $\mathcal{U} = \mathcal{U}(q_1, q_2, \ldots, q_n)$ and

$$\frac{d\mathcal{U}}{dt} = \frac{\partial \mathcal{U}}{\partial q_1} \dot{q}_1 + \ldots \frac{\partial \mathcal{U}}{\partial q_n} \dot{q}_n = \begin{bmatrix} \frac{\partial \mathcal{U}}{\partial q_1} & \ldots & \frac{\partial \mathcal{U}}{\partial q_n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \qquad (24)$$

Since the generalized speeds are always defined as linear combinations of time-derivatives of generalized coordinates, $\dot{q}_1 \ldots \dot{q}_n$ can always by related to $u_1 \ldots u_p$ as

$$\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} = \begin{bmatrix} w_{11} & .. & w_{1p} \\ \vdots & \vdots & \vdots \\ w_{n1} & .. & w_{np} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} + \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad (25)$$

where w_{ij} and x_i (i=1 ... n, j=1 ... p) are functions of $q_1 \ldots q_n$ and time. Substituting equation (25) into equation (24) and subsequently using equation (23) produces

$$\frac{d\mathcal{U}}{dt} = \begin{bmatrix} \frac{\partial \mathcal{U}}{\partial q_1} & \ldots & \frac{\partial \mathcal{U}}{\partial q_n} \end{bmatrix} \left(\begin{bmatrix} w_{11} & .. & w_{1p} \\ \vdots & \vdots & \vdots \\ w_{n1} & .. & w_{np} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} + \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) = -\begin{bmatrix} \mathcal{F}_1 & .. & \mathcal{F}_p \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \qquad (26)$$

Since $u_1 \ldots u_p$ are ***independent***:

$$\begin{bmatrix} \frac{\partial \mathcal{U}}{\partial q_1} & \ldots & \frac{\partial \mathcal{U}}{\partial q_n} \end{bmatrix} \begin{bmatrix} w_{11} & .. & w_{1p} \\ \vdots & \vdots & \vdots \\ w_{n1} & .. & w_{np} \end{bmatrix} \underset{(26)}{=} -\begin{bmatrix} \mathcal{F}_1 & .. & \mathcal{F}_p \end{bmatrix} \qquad (27)$$

$$\begin{bmatrix} \frac{\partial \mathcal{U}}{\partial q_1} & \ldots & \frac{\partial \mathcal{U}}{\partial q_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \underset{(26)}{=} 0 \qquad (28)$$

Since equation (25) shows $w_{ij} = \frac{\partial \dot{q}_i}{\partial u_j}$, equation (27) leads directly to equation (21). When a potential energy U exists, equation (21) can be modified by replacing $\mathcal{U}$ with U. The proof of this is nearly identical, except that one uses the fact that power P and potential energy U are related by $\frac{d\mathrm{U}}{dt} = -\mathrm{P}$.

34.9 Expressions for σ_R and σ for a rigid body

$$\sigma_R^B \;=\; \mathrm{m}^B * \vec{\mathbf{v}}_{\mathrm{R}}^{B_p} \cdot \frac{{}^N\!d\,\vec{\mathbf{v}}_{\mathrm{t}}^{B_p}}{dt} \;+\; \vec{\boldsymbol{\omega}}_{\mathrm{R}} \cdot \vec{\vec{\mathbf{I}}} \cdot \frac{{}^B\!d\,\vec{\boldsymbol{\omega}}_{\mathrm{t}}}{dt} \;+\; \vec{\boldsymbol{\omega}}_{\mathrm{R}} \cdot \left(\vec{\boldsymbol{\omega}}_{\mathrm{t}} \times \vec{\vec{\mathbf{I}}} \cdot \vec{\boldsymbol{\omega}} \right) \tag{29}$$

$$\sigma^B \;=\; \mathrm{m}^B * \vec{\mathbf{v}}^{B_p} \cdot \frac{{}^N\!d\,\vec{\mathbf{v}}_{\mathrm{t}}^{B_p}}{dt} \;+\; \vec{\boldsymbol{\omega}} \cdot \vec{\vec{\mathbf{I}}} \cdot \frac{{}^N\!d\,\vec{\boldsymbol{\omega}}_{\mathrm{t}}}{dt} \tag{30}$$

- m^B is the mass of B and $\vec{\vec{\mathbf{I}}}$ is B's inertia dyadic about B_p.
- B_p is either the mass center of B *or* a point fixed in both B and N. $\vec{\mathbf{v}}^{B_p}$ is B_p's velocity in N.
- $\vec{\mathbf{v}}_{\mathrm{R}}^{B_p}$ is the portion of $\vec{\mathbf{v}}^{B_p}$ that contains $u_1 \dots u_p$ whereas $\vec{\mathbf{v}}_{\mathrm{t}}^{B_p}$ is the portion ***without*** $u_1 \dots u_p$.
- $\vec{\boldsymbol{\omega}}_{\mathrm{R}}$ is the portion of $\vec{\boldsymbol{\omega}}$ (B's angular velocity in N) that contains $u_1 \dots u_p$.
- $\vec{\boldsymbol{\omega}}_{\mathrm{t}}$ is the portion of $\vec{\boldsymbol{\omega}}$ ***without*** $u_1 \dots u_p$.

Optional: Proof of σ_R and σ for a rigid body

To establish the validity of equation (29), start by noting that σ_R^B of a set of β particles $B_1 \dots B_\beta$ of a rigid body B possessing p independent generalized speeds $u_1 \dots u_p$ in a reference frame N is defined as

$$\sigma_R^B \;\underset{(4)}{\triangleq}\; \sum_{i=1}^{\beta} \mathrm{m}^{B_i}\, \vec{\mathbf{v}}_{\mathrm{R}}^{B_i} \cdot \frac{{}^N\!d\,\vec{\mathbf{v}}_{\mathrm{t}}^{B_i}}{dt} \tag{31}$$

- m^{B_i} is the mass of B_i
- $\vec{\mathbf{v}}_{\mathrm{R}}^{B_i}$ is the portion of $\vec{\mathbf{v}}^{B_i}$ (the velocity of B_i in N) that contains $u_1 \dots u_p$
- $\vec{\mathbf{v}}_{\mathrm{t}}^{B_i}$ is the portion of $\vec{\mathbf{v}}^{B_i}$ that does *not* contain $u_1 \dots u_p$
- $\frac{{}^N\!d\,\vec{\mathbf{v}}_{\mathrm{t}}^{B_i}}{dt}$ is the time-derivative in N of $\vec{\mathbf{v}}_{\mathrm{t}}^{B_i}$

It has been shown [36, p. 45] that B's angular velocity in N can always be written in terms of B's partial angular velocity in N for u_r and $\vec{\boldsymbol{\omega}}_{\mathrm{t}}$ (the portion of $\vec{\boldsymbol{\omega}}$ that does *not* contain $u_1 \dots u_p$), as shown in equation (32). By defining $\vec{\boldsymbol{\omega}}_{\mathrm{R}}$ as the portion of $\vec{\boldsymbol{\omega}}$ that contains $u_1 \dots u_p$, i.e., as shown in equation (33), B's angular velocity in N can be written as given in equation (34).

$$\vec{\boldsymbol{\omega}} \;=\; \sum_{r=1}^{p} \frac{\partial\,\vec{\boldsymbol{\omega}}}{\partial u_r} * u_r \;+\; \vec{\boldsymbol{\omega}}_{\mathrm{t}} \tag{32}$$

$$\vec{\boldsymbol{\omega}}_{\mathrm{R}} \;\triangleq\; \sum_{r=1}^{p} \frac{\partial\,\vec{\boldsymbol{\omega}}}{\partial u_r} * u_r \tag{33}$$

$$\vec{\boldsymbol{\omega}} \;=\; \vec{\boldsymbol{\omega}}_{\mathrm{R}} \;+\; \vec{\boldsymbol{\omega}}_{\mathrm{t}} \tag{34}$$

The next step in the proof is to introduce a point B_p that is fixed on B and is *either* the mass center of B *or* a point fixed in N. Since $\vec{\mathbf{v}}^{B_i}$ (the velocity of B_i in N) and $\vec{\mathbf{v}}^{B_p}$ (the velocity of B_p in N) are related by $\vec{\boldsymbol{\omega}}$ and $\vec{\mathbf{r}}_i$ (B_i's position from B_p), as

$$\vec{\mathbf{v}}^{B_i} \;\underset{(10.3)}{=}\; \vec{\mathbf{v}}^{B_p} \;+\; \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_i \tag{35}$$

it can be shown that

$$\vec{\mathbf{v}}_{\mathrm{R}}^{B_i} \;\underset{(34\,35)}{=}\; \vec{\mathbf{v}}_{\mathrm{R}}^{B_p} \;+\; \vec{\boldsymbol{\omega}}_{\mathrm{R}} \times \vec{\mathbf{r}}_i \tag{36}$$

$$\vec{\mathbf{v}}_{\mathrm{t}}^{B_i} \;\underset{(34\,35)}{=}\; \vec{\mathbf{v}}_{\mathrm{t}}^{B_p} \;+\; \vec{\boldsymbol{\omega}}_{\mathrm{t}} \times \vec{\mathbf{r}}_i \tag{37}$$

Time-differentiation of equation (37) in reference frame N gives

$$\frac{{}^N\!d\,\vec{\mathbf{v}}_{\mathrm{t}}^{B_i}}{dt} \;\underset{(37)}{=}\; \frac{{}^N\!d\,\vec{\mathbf{v}}_{\mathrm{t}}^{B_p}}{dt} \;+\; \frac{{}^B\!d\,\vec{\boldsymbol{\omega}}_{\mathrm{t}}}{dt} \times \vec{\mathbf{r}}_i \;+\; \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}}_{\mathrm{t}} \times \vec{\mathbf{r}}_i) \tag{38}$$

Substituting equations (36) and (38) into equation (31) gives

$$\sigma_R^B \underset{(31)}{=} \sum_{i=1}^{\beta} \text{m}^{B_i} \underbrace{\left(\vec{\mathbf{v}}_{\text{R}}^{B_p} + \vec{\boldsymbol{\omega}}_{\text{R}} \times \vec{\mathbf{r}}_i\right)}_{(36)} \cdot \underbrace{\left[\frac{{}^N d\,\vec{\mathbf{v}}_{\text{t}}^{B_p}}{dt} + \frac{{}^B d\,\vec{\boldsymbol{\omega}}_{\text{t}}}{dt} \times \vec{\mathbf{r}}_i + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}}_{\text{t}} \times \vec{\mathbf{r}}_i)\right]}_{(38)} \tag{39}$$

Distributing the dot-product and making use of the fact that since B_p is *either* the mass center of B *or* a point fixed in N,[5]

$$\sum_{i=1}^{\beta} \text{m}^{B_i}\,\vec{\mathbf{r}}_i = \vec{\mathbf{0}} \qquad \text{or} \qquad \vec{\mathbf{v}}_{\text{R}}^{B_p} = \vec{\mathbf{0}} \tag{40}$$

and using the fact that $\text{m}^B \triangleq \sum_{i=1}^{\beta} \text{m}^{B_i}$ and that for any vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, $\vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$, leads to

$$\sigma_R^B \underset{(39\,40)}{=} \text{m}^B\,\vec{\mathbf{v}}_{\text{R}}^{B_p} \cdot \frac{{}^N d\,\vec{\mathbf{v}}_{\text{t}}^{B_p}}{dt} + \vec{\boldsymbol{\omega}}_{\text{R}} \cdot \sum_{i=1}^{\beta} \text{m}^{B_i}\,\vec{\mathbf{r}}_i \times \left(\frac{{}^B d\,\vec{\boldsymbol{\omega}}_{\text{t}}}{dt} \times \vec{\mathbf{r}}_i\right) + \vec{\boldsymbol{\omega}}_{\text{R}} \cdot \sum_{i=1}^{\beta} \text{m}^{B_i}\,\vec{\mathbf{r}}_i \times [\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}}_{\text{t}} \times \vec{\mathbf{r}}_i)] \tag{41}$$

Focusing attention on the summation in the second term on the right-hand side of equation (41) and making use of the vector identity $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} + \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$ and then the definition[6] of the inertia dyadic of B about B_p, one finds

$$\begin{aligned}
\sum_{i=1}^{\beta} \text{m}^{B_i}\,\vec{\mathbf{r}}_i \times \left(\frac{{}^B d\,\vec{\boldsymbol{\omega}}_{\text{t}}}{dt} \times \vec{\mathbf{r}}_i\right) &= \sum_{i=1}^{\beta} \text{m}^{B_i} \left[(\vec{\mathbf{r}}_i \cdot \vec{\mathbf{r}}_i)\,\frac{{}^B d\,\vec{\boldsymbol{\omega}}_{\text{t}}}{dt} - \vec{\mathbf{r}}_i\left(\vec{\mathbf{r}}_i \cdot \frac{{}^B d\,\vec{\boldsymbol{\omega}}_{\text{t}}}{dt}\right)\right] \\
&= \left\{\sum_{i=1}^{\beta} \text{m}^{B_i} \left[(\vec{\mathbf{r}}_i \cdot \vec{\mathbf{r}}_i) * \vec{\vec{\mathbf{1}}} - \vec{\mathbf{r}}_i * \vec{\mathbf{r}}_i\right]\right\} \cdot \frac{{}^B d\,\vec{\boldsymbol{\omega}}_{\text{t}}}{dt} = \vec{\vec{\mathbf{I}}} \cdot \frac{{}^B d\,\vec{\boldsymbol{\omega}}_{\text{t}}}{dt}
\end{aligned} \tag{42}$$

Focusing attention on the summation in the third term on the right-hand side of equation (41) and again making use of the vector identity $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} + \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$ and then a relationship[7] between the summation $\sum_{i=1}^{n} \text{m}^{Q_i}\,\vec{\mathbf{r}}_i * \vec{\mathbf{r}}_i$ and the inertia dyadic of B about B_p, one finds

$$\begin{aligned}
\sum_{i=1}^{\beta} \text{m}^{B_i}\,\vec{\mathbf{r}}_i \times [\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}}_{\text{t}} \times \vec{\mathbf{r}}_i)] &= \sum_{i=1}^{\beta} \text{m}^{B_i}\,\vec{\mathbf{r}}_i \times [\vec{\boldsymbol{\omega}}_{\text{t}} * (\vec{\boldsymbol{\omega}} \cdot \vec{\mathbf{r}}_i) - \vec{\mathbf{r}}_i * (\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{\omega}}_{\text{t}})] \\
&= \vec{\boldsymbol{\omega}} \cdot \left(\sum_{i=1}^{\beta} \text{m}^{B_i}\,\vec{\mathbf{r}}_i * \vec{\mathbf{r}}_i\right) \times \vec{\boldsymbol{\omega}}_{\text{t}} \\
&\underset{(16.15)}{=} \vec{\boldsymbol{\omega}} \cdot \left[\frac{1}{2}\,\text{trace}(\vec{\vec{\mathbf{I}}}) * \vec{\vec{\mathbf{1}}} - \vec{\vec{\mathbf{I}}}\right] \times \vec{\boldsymbol{\omega}}_{\text{t}} \\
&= \frac{1}{2}\,\text{trace}(\vec{\vec{\mathbf{I}}}) * \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{\omega}}_{\text{t}} - \vec{\boldsymbol{\omega}} \cdot \vec{\vec{\mathbf{I}}} \times \vec{\boldsymbol{\omega}}_{\text{t}} \\
&\underset{(34)}{=} \frac{1}{2}\,\text{trace}(\vec{\vec{\mathbf{I}}}) * \vec{\boldsymbol{\omega}}_{\text{R}} \times \vec{\boldsymbol{\omega}}_{\text{t}} - \vec{\boldsymbol{\omega}} \cdot \vec{\vec{\mathbf{I}}} \times \vec{\boldsymbol{\omega}}_{\text{t}}
\end{aligned} \tag{43}$$

Pre-dot multiplication of equation (43) with $\vec{\boldsymbol{\omega}}_{\text{R}}$ gives

$$\vec{\boldsymbol{\omega}}_{\text{R}} \cdot \sum_{i=1}^{\beta} \text{m}^{B_i}\,\vec{\mathbf{r}}_i \times [\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}}_{\text{t}} \times \vec{\mathbf{r}}_i)] \underset{(43)}{=} -\vec{\boldsymbol{\omega}}_{\text{R}} \cdot \left(\vec{\boldsymbol{\omega}} \cdot \vec{\vec{\mathbf{I}}} \times \vec{\boldsymbol{\omega}}_{\text{t}}\right) = \vec{\boldsymbol{\omega}}_{\text{R}} \cdot \left(\vec{\boldsymbol{\omega}}_{\text{t}} \times \vec{\vec{\mathbf{I}}} \cdot \vec{\boldsymbol{\omega}}\right) \tag{44}$$

Substitution of equation (42) into the second term on the right-hand side of equation (41) and substitution of equation (44) for the third term on the right-hand side of equation (41) produces equation (29).

[5] The first relationship in equation (40) is the definition of the center of mass of B.

[6] The definition of the inertia dyadic of B about B_p is $\vec{\vec{\mathbf{I}}} \triangleq \sum_{i=1}^{\beta} \text{m}^{B_i} \left[(\vec{\mathbf{r}}_i \cdot \vec{\mathbf{r}}_i) * \vec{\vec{\mathbf{1}}} - \vec{\mathbf{r}}_i * \vec{\mathbf{r}}_i\right]$

[7] The relationship between the summation and the inertia dyadic used here is $\sum_{i=1}^{n} \text{m}^{Q_i}\,\vec{\mathbf{r}}_i * \vec{\mathbf{r}}_i = \frac{1}{2}\,\text{trace}(\vec{\vec{\mathbf{I}}}) * \vec{\vec{\mathbf{1}}} - \vec{\vec{\mathbf{I}}}$.

34.10 Optional: Proof of the generalized energy integral

To establish the validity of equation (2), start by noting that the law of motion relates $\vec{\mathbf{F}}^{Q_i}$ (the resultant of all forces on a particle Q_i) with m^{Q_i} (the mass of Q_i) and $\vec{\mathbf{a}}^{Q_i}$ (the acceleration of Q_i in a Newtonian reference frame N), by

$$\vec{\mathbf{F}}^{Q_i} \;=\; \mathrm{m}^{Q_i} * \vec{\mathbf{a}}^{Q_i} \tag{45}$$

Dot-multiplication of both sides of equation (45) with $\vec{\mathbf{v}}_{\mathrm{R}}^{Q_i}$ gives

$$\vec{\mathbf{F}}^{Q_i} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \underset{(45)}{=} \mathrm{m}^{Q_i} * \vec{\mathbf{a}}^{Q_i} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \tag{46}$$

The definitions of $\mathcal{P}^{Q_i}$ (the generalized power of Q_i in N) and $\vec{\mathbf{a}}^{Q_i}$ (the acceleration of Q_i in N) are

$$\mathcal{P}^{Q_i} \underset{(14)}{\triangleq} \vec{\mathbf{F}}^{Q_i} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \qquad\qquad \vec{\mathbf{a}}^{Q_i} \triangleq \frac{{}^{N}\!d\, \vec{\mathbf{v}}^{Q_i}}{dt} \tag{47}$$

Hence, equation (46) can be re-expressed as

$$\mathcal{P}^{Q_i} \underset{(46\,47)}{=} \mathrm{m}^{Q_i} * \frac{{}^{N}\!d\, \vec{\mathbf{v}}^{Q_i}}{dt} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \tag{48}$$

Equation (12) showed $\vec{\mathbf{v}}^{Q_i}$ can be expressed as $\vec{\mathbf{v}}^{Q_i} = \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} + \vec{\mathbf{v}}_{\mathrm{t}}^{Q_i}$ so equation (48) can be rearranged to

$$\mathcal{P}^{Q_i} \underset{(48\,12)}{=} \mathrm{m}^{Q_i} * \frac{{}^{N}\!d\, \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i}}{dt} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \;+\; \mathrm{m}^{Q_i} * \frac{{}^{N}\!d\, \vec{\mathbf{v}}_{\mathrm{t}}^{Q_i}}{dt} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \tag{49}$$

To show that the first term on the right-hand side of equation (49) is the time-derivative of $\mathrm{K}_2^{Q_i}$ (the kinetic energy of Q_i in N of degree 2 in $u_1 \dots u_p$), note that the definition of $\mathrm{K}_2^{Q_i}$ is

$$\mathrm{K}_2^{Q_i} \underset{(13)}{\triangleq} \frac{1}{2}\, \mathrm{m}^{Q_i} * \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \tag{50}$$

Time-differentiation of both sides of equation (50) leads to

$$\frac{d\, \mathrm{K}_2^{Q_i}}{dt} \underset{(50)}{=} \mathrm{m}^{Q_i} * \frac{{}^{N}\!d\, \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i}}{dt} \cdot \vec{\mathbf{v}}_{\mathrm{R}}^{Q_i} \tag{51}$$

Since the first term on the right-hand side of equation (49) is identical to the right-hand side of equation (51) and since the second term on the right-hand side of equation (49) is by definition [see equation (4)] $\sigma_R^{Q_i}$, equation (49) can be re-expressed as

$$\mathcal{P}^{Q_i} \underset{(49)}{=} \underset{(51)}{\frac{d\, \mathrm{K}_2^{Q_i}}{dt}} \;+\; \underset{(4)}{\sigma_R^{Q_i}} \tag{52}$$

When a system S consists of ν particles $Q_1 \dots Q_\nu$, equation (52) can be applied to each particle and the resulting set of equations can be summed, yielding

$$\sum_{i=1}^{\nu} \mathcal{P}^{Q_i} \underset{(52)}{=} \sum_{i=1}^{\nu} \frac{d\, \mathrm{K}_2^{Q_i}}{dt} \;+\; \sum_{i=1}^{\nu} \sigma_R^{Q_i} \tag{53}$$

Interchanging the derivative and summation on the right-hand side of equation (53) produces

$$\sum_{i=1}^{n} {}^{N}\mathcal{P}^{Q_i} \underset{(53)}{=} \frac{d}{dt}\left(\sum_{i=1}^{n} \mathrm{K}_2^{Q_i} \right) \;+\; \sum_{i=1}^{\nu} \sigma_R^{Q_i} \tag{54}$$

Since $\mathcal{P}$ (the generalized power of S in N), K_2 (the kinetic energy of S in N), and σ_R are defined as

$$\mathcal{P} \underset{(15)}{\triangleq} \sum_{i=1}^{\nu} \mathcal{P}^{Q_i} \qquad \mathrm{K} \underset{(13)}{\triangleq} \sum_{i=1}^{\nu} \mathrm{K}^{Q_i} \qquad \sigma_R \underset{(4)}{\triangleq} \sum_{i=1}^{\nu} \sigma_R^{Q_i} \tag{55}$$

equation (54) may be rewritten as

$$\mathcal{P} \underset{(54\,55)}{=} \frac{d\mathrm{K}_2}{dt} + \sigma_R \tag{56}$$

Time-integration of equation (56) and subsequent rearrangement gives

$$\mathcal{E}_Z \underset{(56)}{=} \mathrm{K}_2 + \int \sigma_R * dt - \int \mathcal{P} * dt \tag{57}$$

where $\mathcal{E}_Z$ is an arbitrary constant of integration having units of energy. Combining the information in equation (16) and equation (20), leads to

$$\mathcal{E}_Z \underset{(57)}{=} \mathrm{K}_2 + \int \sigma_R * dt + \mathcal{U} - \int \mathcal{P}_{nonconservative} * dt \tag{58}$$

Defining Z as was done in equation (3) leads directly to equation (2).