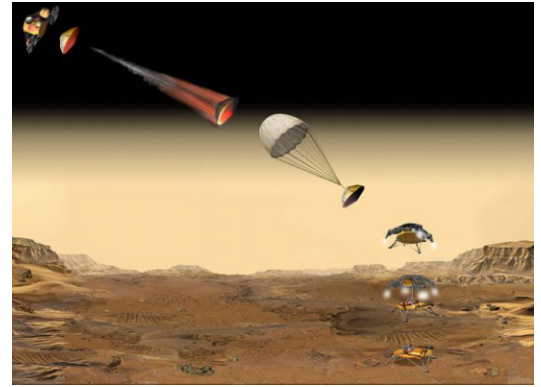


Chapter 2

Vectors



Courtesy NASA/JPL-Caltech

Summary (see examples in Hw 1, 2, 3)

Circa 1900 A.D., J. Williard Gibbs invented a useful combination of magnitude and direction called **vectors** and their higher-dimensional counterparts **dyadics**, **triadics**, and **polyadics**. Vectors are an important **geometrical tool** e.g., for surveying, motion analysis, lasers, optics, computer graphics, animation, CAD/CAE (computer aided drawing/engineering), and FEA.

Symbol	Description	Details
$\vec{0}, \hat{u}$	Zero vector and unit vector.	Sections 2.3, 2.4
$+ - *$	Vector addition, negation, subtraction, and multiplication/division with a scalar.	Sections 2.6 - 2.8
$\cdot \times$	Vector dot product and cross product.	Sections 2.9, 2.10
$\frac{F}{dt}$	Vector differentiation.	Chapters 6, 7



2.1 Examples of scalars, vectors, and dyadics

- A **scalar** is a non-directional quantity (e.g., a real number). Examples include:

time	density	volume	mass	moment of inertia	temperature
distance	speed	angle	weight	potential energy	kinetic energy

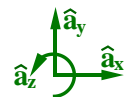
- A **vector** is a quantity that has magnitude and **one** associated direction. For example, a **velocity vector** has speed (how fast something is moving) and direction (which way it is going). A **force vector** has magnitude (how hard something is pushed) and direction (which way it is shoved). Examples include:

position vector	velocity	acceleration	translational momentum	force
impulse	angular velocity	angular acceleration	angular momentum	torque

- A **dyad** is a quantity with magnitude and **two** associated directions. For example, **stress** associates with area and force (both regarded as vectors). A **dyadic** is the **sum of dyads**. For example, an **inertia dyadic** (Chapter 16) is the sum of dyads associated with moments and products of inertia.
- A **triad** is a quantity that has magnitude and **three** directions. A **triadic** is the sum of triads.

Words: Vector and column matrices. Although mathematics uses the word¹ “vector” to describe a column matrix, a column matrix does **not** have direction. To associate direction, attach a basis e.g., as shown below.

$$\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z = \begin{bmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\hat{a}_{xyz}} \quad \text{where } \hat{a}_x, \hat{a}_y, \hat{a}_z \text{ are orthogonal unit vectors}$$

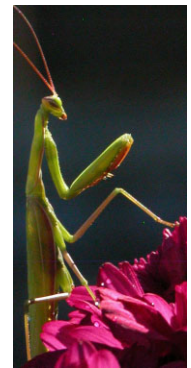


Note: Although it can be helpful to represent vectors with orthogonal unit vectors (e.g., $\hat{x}, \hat{y}, \hat{z}$ or $\hat{i}, \hat{j}, \hat{k}$), it is not always necessary, desirable, or efficient. Postponing resolution of vectors into components allows maximum use of simplifying vector properties and avoids simplifications such as $\sin^2(\theta) + \cos^2(\theta) = 1$ (see Homework 2.9).

¹Words have context. Some words are contronyms (opposite meanings) such as “fast” and “bolt” (move quickly or fasten).

2.2 Definition of a vector

A *vector* is defined as a quantity having *magnitude* and *direction*.^a Vectors are represented graphically with straight or curved arrows (examples below).



Courtesy Bro. Claude Rheume LaSalette.

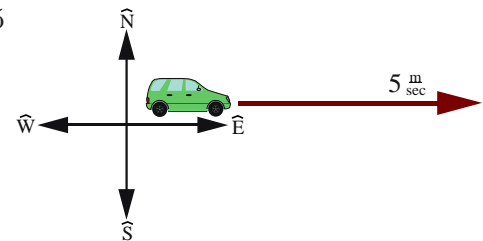
Certain vectors have additional special properties. For example, a *position vector* is associated with two points and has units of distance. A *bound vector* such as *force* is associated with a point (or line of action).

^a A vector's *magnitude* is a real non-negative number. A vector's *direction* can be resolved into *orientation* and *sense*. For example, a highway has an orientation (e.g., east-west) and a vehicle traveling east has a sense. Knowing both the orientation of a line and the sense on the line gives direction. Changing a vector's orientation or sense changes its direction.

Example of a vector: Consider the traffic report “the vehicle is heading East at $5 \frac{\text{m}}{\text{s}}$ ”. It is *convenient* to name these two pieces of information (speed and direction) as a “*velocity vector*” and represent them mathematically as $5 * \vec{\text{East}}$ (direction is identified with an arrow and/or bold-face font or with a hat for a *unit vector* such as $\hat{\text{East}}$). The vehicle's speed is always a real non-negative number, equal to the *magnitude* of the velocity vector. The combination of *magnitude* and *direction* is a *vector*.

For example, the vector \vec{v} describing a vehicle traveling with speed 5 to the East is graphically depicted to the right, and is written

$$\vec{v} = 5 * \hat{\text{East}} \quad \text{or} \quad \vec{v} = 5 \hat{\text{East}}$$



A vehicle traveling with speed 5 to the West is

$$\vec{v} = 5 \hat{\text{West}} \quad \text{or} \quad \vec{v} = -5 \hat{\text{East}}$$

Note: The negative sign in $-5 \hat{\text{East}}$ is associated with the vector's direction (the vector's magnitude is inherently non-negative). When a vector is written in terms of a scalar x that can be **positive** or zero or **negative**, e.g., as $x \hat{\text{East}}$, x is called the *East measure* of the vector, whereas the vector's non-negative *magnitude* is $\text{abs}(x)$.

2.3 Zero vector $\vec{0}$ and its properties

A *zero vector* $\vec{0}$ is defined as a vector whose magnitude is zero.²

Addition of a vector \vec{v} with a zero vector:	$\vec{v} + \vec{0} = \vec{v}$	
Dot product with a zero vector:	$\vec{v} \cdot \vec{0} = 0$	$\vec{0}$ is <i>perpendicular</i> to all vectors
Cross product with a zero vector:	$\vec{v} \times \vec{0} = \vec{0}$	$\vec{0}$ is <i>parallel</i> to all vectors
Derivative of the zero vector:	$\frac{F d \vec{0}}{dt} = \vec{0}$	F is any reference frame

Vectors \vec{a} and \vec{b} are said to be “*perpendicular*” if $\vec{a} \cdot \vec{b} = 0$ whereas \vec{a} and \vec{b} are “*parallel*” if $\vec{a} \times \vec{b} = \vec{0}$.

Note: Some say \vec{a} and \vec{b} are “*parallel*” only if \vec{a} and \vec{b} have the same direction and anti-parallel if \vec{a} and \vec{b} have opposite directions.

²The direction of a zero vector $\vec{0}$ is arbitrary and may be regarded as having **any** direction so that $\vec{0}$ is *parallel* to all vectors, $\vec{0}$ is *perpendicular* to all vectors, all zero vectors are equal, and one may use the definite pronoun “the” instead of the indefinite “a” e.g., “the zero vector”. It is improper to say the *zero vector* has no direction as a vector is **defined** to have both magnitude and direction. It is also improper to say a *zero vector* has all directions as a vector is defined to have a magnitude and **a** direction (as contrasted with a dyad which has 2 directions or triad which has 3 directions).

2.4 Unit vectors

A **unit vector** is defined as a vector whose magnitude is 1, and is designated with a special hat, e.g., $\hat{\mathbf{u}}$.

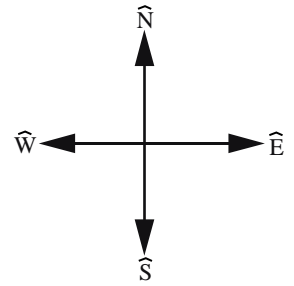
Unit vectors can be “**sign posts**”, e.g., unit vectors $\hat{\mathbf{N}}$, $\hat{\mathbf{S}}$, $\hat{\mathbf{W}}$, $\hat{\mathbf{E}}$ associated with local Earth directions **North**, **South**, **West**, **East**, respectively.

The direction of unit vectors are chosen to simplify communication and to produce efficient equations. Other useful “sign posts” are:

- Unit vector directed from one point to another point
- Unit vector directed locally vertical
- Unit vector parallel to the edge of an object
- Unit vector tangent to a curve or perpendicular to a surface

A unit vector can be defined so it has the same direction as an arbitrary non-zero vector $\vec{\mathbf{v}}$ by dividing $\vec{\mathbf{v}}$ by $|\vec{\mathbf{v}}|$ (the magnitude of $\vec{\mathbf{v}}$).

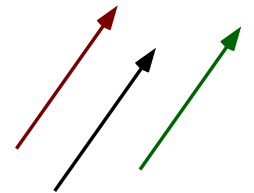
To avoid divide-by-zero problems during numerical computation, approximate the unit vector with a “small” positive real number ϵ in the denominator.



$$\text{unit Vector} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \approx \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}| + \epsilon} \quad (1)$$

2.5 Equal vectors (=)

Two vectors are “equal” when they have the same magnitude and same direction.^a Shown to the right are three **equal vectors**. Although each has a different location, the vectors are equal because they have the same magnitude and direction.^b

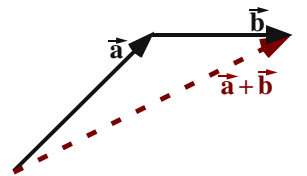
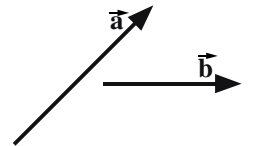


^aHomework 2.6 draws vectors of different magnitude, **orientation**, and **sense**.

^bSome vectors have additional properties. For example, a position vector is associated with two points. Two position vectors are **equal position vectors** when they have the same magnitude, same direction, and are associated with the same points. Two force vectors are **equal force vectors** when they have the same magnitude, direction, and point of application.

2.6 Vector addition (+)

As graphically shown to the right, adding two vectors $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ produces a vector.^a First, vector $\vec{\mathbf{b}}$ is translated^b so its tail is at the tip of $\vec{\mathbf{a}}$. Next, the vector $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ is drawn from the tail of $\vec{\mathbf{a}}$ to the tip of the translated $\vec{\mathbf{b}}$.



Properties of vector addition

Commutative law: $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}}$

Associative law: $(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}$

Addition of zero vector: $\vec{\mathbf{a}} + \vec{\mathbf{0}} = \vec{\mathbf{a}}$

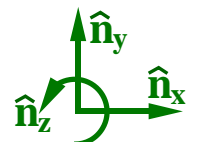
^aIt does not make sense to add vectors with different units, e.g., it is nonsensical to add a velocity vector with units of $\frac{\text{m}}{\text{s}}$ with an angular velocity vector with units of $\frac{\text{rad}}{\text{sec}}$.

^bTranslating $\vec{\mathbf{b}}$ does *not* change the magnitude or direction of $\vec{\mathbf{b}}$, and so produces an equal $\vec{\mathbf{b}}$.

Example: Vector addition (+)

Shown to the right is an example of how to add vector $\vec{\mathbf{w}}$ to vector $\vec{\mathbf{v}}$, each which is expressed in terms of orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$.

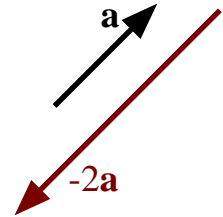
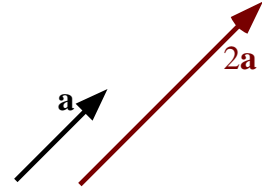
$$\begin{aligned} \vec{\mathbf{v}} &= 7\hat{\mathbf{n}}_x + 5\hat{\mathbf{n}}_y + 4\hat{\mathbf{n}}_z \\ + \vec{\mathbf{w}} &= 2\hat{\mathbf{n}}_x + 3\hat{\mathbf{n}}_y + 2\hat{\mathbf{n}}_z \\ \hline &= 9\hat{\mathbf{n}}_x + 8\hat{\mathbf{n}}_y + 6\hat{\mathbf{n}}_z \end{aligned}$$



2.7 Vector multiplied or divided by a scalar (* or /)

To the right is a graphical representation of multiplying a vector \vec{a} by a scalar.^a

- Multiplying a vector by a **positive** number (other than 1) changes the vector's magnitude.
- Multiplying a vector by a **negative** number changes the vector's magnitude **and** reverses the **sense** of the vector.
- Dividing a vector \vec{a} by a scalar s_1 is defined as $\frac{\vec{a}}{s_1} \triangleq \frac{1}{s_1} * \vec{a}$



Properties of multiplication of a vector by a scalar s_1 or s_2

- Commutative law: $s_1 \vec{a} = \vec{a} s_1$
 Associative law: $s_1 (s_2 \vec{a}) = (s_1 s_2) \vec{a} = s_2 (s_1 \vec{a}) = s_1 s_2 \vec{a}$
 Distributive law: $(s_1 + s_2) \vec{a} = s_1 \vec{a} + s_2 \vec{a}$
 Distributive law: $s_1 (\vec{a} + \vec{b}) = s_1 \vec{a} + s_1 \vec{b}$
 Multiplication by zero: $0 * \vec{a} = \vec{0}$

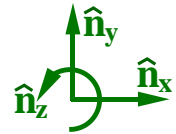
^aHomework 2.9 multiplies a vector \vec{b} by various scalars.

Example: Vector scalar multiplication and division (* and /)

Given: $\vec{v} = 7\hat{n}_x + 5\hat{n}_y + 4\hat{n}_z \Rightarrow$

$$5\vec{v} = 35\hat{n}_x + 25\hat{n}_y + 20\hat{n}_z$$

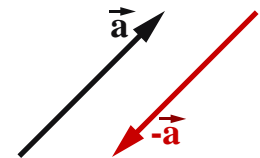
$$\frac{\vec{v}}{-2} = -3.5\hat{n}_x - 2.5\hat{n}_y - 2\hat{n}_z$$



2.8 Vector negation and subtraction (-)

Negation: A graphical representation of negating a vector \vec{a} is shown right.

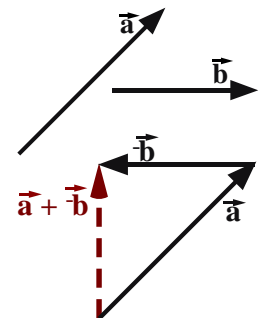
Negating a vector (multiplying the vector by -1) changes the **sense** of a vector without changing its magnitude or orientation. In other words, multiplying a vector by -1 reverses the sense of the vector (it points in the opposite direction).



Subtraction: As drawn right, the process of subtracting a vector \vec{b} from a vector \vec{a} is simply addition and negation.^a

$$\vec{a} - \vec{b} \triangleq \vec{a} + -\vec{b}$$

After negating vector \vec{b} , it is translated so the tail of $-\vec{b}$ is at the tip of \vec{a} . Next, vector $\vec{a} + -\vec{b}$ is drawn from the tail of \vec{a} to the tip of the translated $-\vec{b}$.

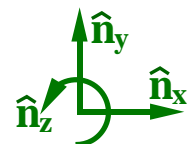


^aIn most (or all) mathematical processes, subtraction is defined as negation and addition.

Example: Vector subtraction (-)

Shown right is an example of how to subtract vector \vec{w} from vector \vec{v} , each which is expressed in terms of orthogonal unit vectors \hat{n}_x , \hat{n}_y , \hat{n}_z .

$$\begin{array}{r} \vec{v} = 7\hat{n}_x + 5\hat{n}_y + 4\hat{n}_z \\ - \vec{w} = 2\hat{n}_x + 3\hat{n}_y + 2\hat{n}_z \\ \hline 5\hat{n}_x + 2\hat{n}_y + 2\hat{n}_z \end{array}$$



2.9 Vector dot product (\cdot)

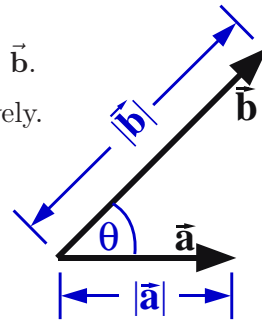
Equation (2) defines the **dot product** of vectors \vec{a} and \vec{b} .

- $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of \vec{a} and \vec{b} , respectively.
- θ is the smallest angle between \vec{a} and \vec{b} ($0 \leq \theta \leq \pi$).

Equation (3) is a rearrangement of equation (2) that is useful for calculating the angle θ between two vectors.

Note: \vec{a} and \vec{b} are “**perpendicular**” when $\vec{a} \cdot \vec{b} = 0$.

Note: Dot-products encapsulate the **law of cosines**.



$$\vec{a} \cdot \vec{b} \triangleq |\vec{a}| |\vec{b}| \cos(\theta) \quad (2)$$

$$\cos(\theta) \stackrel{(2)}{=} \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \quad (3)$$

Use **acos** to calculate θ .

Equation (2) shows $\vec{v} \cdot \vec{v} = |\vec{v}|^2$. Hence, the dot product can calculate a vector’s **magnitude** as shown for $|\vec{v}|$ in equation (4).

Equation (4) also defines **vector exponentiation** \vec{v}^n (vector \vec{v} raised to scalar power n) as a non-negative scalar.

Example: Kinetic energy $K = \frac{1}{2} m \vec{v}^2 \stackrel{(4)}{=} \frac{1}{2} m \vec{v} \cdot \vec{v}$

$$\begin{aligned} \vec{v}^2 &\triangleq |\vec{v}|^2 = \vec{v} \cdot \vec{v} \\ |\vec{v}| &= +\sqrt{\vec{v} \cdot \vec{v}} \\ \vec{v}^n &\triangleq |\vec{v}|^n = +(\vec{v} \cdot \vec{v})^{\frac{n}{2}} \end{aligned} \quad (4)$$

2.9.1 Properties of the dot-product (\cdot)

Dot product with a zero vector	$\vec{a} \cdot \vec{0} = 0$
Dot product of perpendicular vectors	$\vec{a} \cdot \vec{b} = 0$ if $\vec{a} \perp \vec{b}$
Dot product of parallel vectors	$\vec{a} \cdot \vec{b} = \pm \vec{a} \vec{b} $ if $\vec{a} \parallel \vec{b}$
Dot product with vectors scaled by s_1 and s_2	$s_1 \vec{a} \cdot s_2 \vec{b} = s_1 s_2 (\vec{a} \cdot \vec{b})$
Commutative law	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
Distributive law	$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
Distributive law	$(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$

Note: The distributive law for dot-products and cross-products is proved in [36, pgs. 23-24, 32-34].

2.9.2 Uses for the dot-product (\cdot)

- Calculating an **angle** between two vectors [see equation (3) and example in Section 3.3] or determining when two vectors are **perpendicular**, e.g., $\vec{a} \cdot \vec{b} = 0$.
- Calculating a vector’s **magnitude** [see equation (4) and **distance** examples in Sections 3.2 and 3.3].
- Changing a **vector equation** into a **scalar equation** (see Homework 2.31).
- Calculating a **unit vector** in the direction of a vector \vec{v} [see equation (1)]

$$\text{unitVector} \stackrel{(1)}{=} \frac{\vec{v}}{|\vec{v}|}$$

- **Projection** of a vector \vec{v} in the direction of \vec{b} is defined:

$$\frac{\vec{v} \cdot \vec{b}}{|\vec{b}|}$$

See Section 4.6 for **projections, measures, coefficients, components**.

Projection of \vec{v} onto the plane N perpendicular to \hat{n} : $\vec{v}_N = \vec{v} - (\vec{v} \cdot \hat{n}) \hat{n} = \hat{n} \times (\vec{v} \times \hat{n})$.

Context: \vec{v} is a vector “bound” to a point v_o whose position vector \vec{r} from a point N_o fixed in N has $\vec{r} \cdot \hat{n} > 0$.

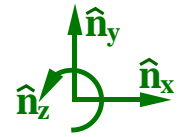
Example: Projection of a position vector \vec{r} (from N_o to a point R , where $\vec{r} \cdot \hat{n} > 0$) onto N : $\vec{r} - (\vec{r} \cdot \hat{n}) \hat{n}$.

Projection of a parallelogram with edges characterized by \vec{p} and \vec{q} onto plane N : $|\vec{p}_N \times \vec{q}_N| \hat{n}$.
Magnitude of \vec{p} ’s projection on N crossed-with \vec{q} ’s projection on N times \hat{n} : $|\vec{p} \times \vec{q} + [(\vec{p} \cdot \hat{n}) \vec{q} - (\vec{q} \cdot \hat{n}) \vec{p}] \times \hat{n}| \hat{n}$.

2.9.3 Special case: Dot-products with orthogonal unit vectors

When $\hat{n}_x, \hat{n}_y, \hat{n}_z$ are *orthogonal unit* vectors, it can be shown (see Homework 2.4)

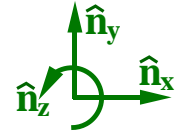
$$(a_x \hat{n}_x + a_y \hat{n}_y + a_z \hat{n}_z) \cdot (b_x \hat{n}_x + b_y \hat{n}_y + b_z \hat{n}_z) = a_x b_x + a_y b_y + a_z b_z$$



Optional: Special case of dot product as matrix multiplication

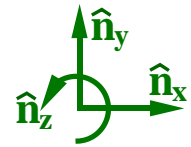
When one defines $\vec{a} \triangleq a_x \hat{n}_x + a_y \hat{n}_y + a_z \hat{n}_z$ and $\vec{b} \triangleq b_x \hat{n}_x + b_y \hat{n}_y + b_z \hat{n}_z$ in terms of the *orthogonal unit* vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$, the dot-product $\vec{a} \cdot \vec{b}$ is related to the multiplication of the $\hat{n}_x, \hat{n}_y, \hat{n}_z$ row matrix representation of \vec{a} with the $\hat{n}_x, \hat{n}_y, \hat{n}_z$ column matrix representation of \vec{b} as

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}_{\hat{n}_{xyz}} * \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}_{\hat{n}_{xyz}} = \begin{bmatrix} a_x b_x + a_y b_y + a_z b_z \end{bmatrix}$$



2.9.4 Examples: Vector dot-products (\cdot)

The following shows how to use dot-products with the vectors \vec{v} and \vec{w} , each which is expressed in terms of the orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$ shown to the right.



$$\vec{v} = 7\hat{n}_x + 5\hat{n}_y + 4\hat{n}_z \quad \vec{w} = 2\hat{n}_x + 3\hat{n}_y + 2\hat{n}_z$$

\hat{n}_x measure of \vec{v}

$$\vec{v} \cdot \hat{n}_x = 7 \quad (\text{measures how much of } \vec{v} \text{ is in the } \hat{n}_x \text{ direction})$$

$$\vec{v} \cdot \vec{v} = 7^2 + 5^2 + 4^2 = 90$$

$$|\vec{v}| = \sqrt{90} \approx 9.4868$$

$$\vec{w} \cdot \vec{w} = 2^2 + 3^2 + 2^2 = 17$$

$$|\vec{w}| = \sqrt{17} \approx 4.1231$$

Unit vector in the direction of \vec{v} :

$$\frac{\vec{v}}{|\vec{v}|} = \frac{7\hat{n}_x + 5\hat{n}_y + 4\hat{n}_z}{\sqrt{90}} \approx 0.738\hat{n}_x + 0.527\hat{n}_y + 0.422\hat{n}_z$$

Unit vector in the direction of \vec{w} :

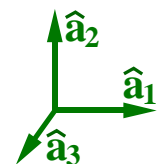
$$\frac{\vec{w}}{|\vec{w}|} = \frac{2\hat{n}_x + 3\hat{n}_y + 2\hat{n}_z}{\sqrt{17}} \approx 0.485\hat{n}_x + 0.728\hat{n}_y + 0.485\hat{n}_z$$

$$\vec{v} \cdot \vec{w} = 7*2 + 5*3 + 4*2 = 37 \quad \angle(\vec{v}, \vec{w}) = \text{acos}\left(\frac{37}{\sqrt{90}\sqrt{17}}\right) \approx 0.33 \text{ rad} \approx 18.9^\circ$$

2.9.5 Dot-products to change vector equations to scalar equations (see Hw 1.31)

One way to form up to three linearly independent scalar equations from the vector equation $\vec{v} = \vec{0}$ is by dot-multiplying $\vec{v} = \vec{0}$ with three orthogonal unit vectors $\hat{a}_1, \hat{a}_2, \hat{a}_3$, i.e.,

Method 1: if $\vec{v} = \vec{0} \Rightarrow \boxed{\vec{v} \cdot \hat{a}_1 = 0 \quad \vec{v} \cdot \hat{a}_2 = 0 \quad \vec{v} \cdot \hat{a}_3 = 0}$



Section 2.11.2 describes another way to form three *different* scalar equations from $\vec{v} = \vec{0}$.



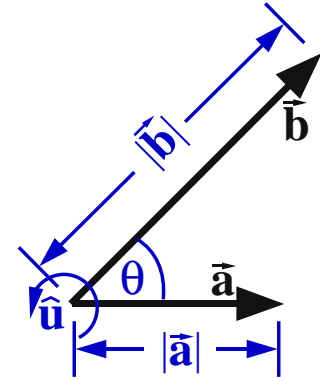
Courtesy Accuray Inc.. Dot-products are heavily used in radiation and other medical equipment.

2.10 Vector cross product (\times)

The **cross product** of a vector \vec{a} with a vector \vec{b} is defined in equation (5).

- $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of \vec{a} and \vec{b} , respectively
- θ is the smallest angle between \vec{a} and \vec{b} ($0 \leq \theta \leq \pi$).
- \hat{u} is the unit vector **perpendicular** to both \vec{a} and \vec{b} .
The direction of \hat{u} is determined by the **right-hand rule**.^a

Note: $|\vec{a}| |\vec{b}| \sin(\theta)$ [the coefficient of \hat{u} in equation (5)] is inherently **non-negative** because $\sin(\theta) \geq 0$ since $0 \leq \theta \leq \pi$. Hence, $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin(\theta)$.



$$\vec{a} \times \vec{b} \triangleq |\vec{a}| |\vec{b}| \sin(\theta) \hat{u} \quad (5)$$

^aThe right-hand rule is a convention, much like driving on the right-hand side of the road in North America. Until 1965, the Soviet Union used the left-hand rule.

2.10.1 Properties of the cross-product (\times)

Cross product with a zero vector

$$\vec{a} \times \vec{0} = \vec{0}$$

Cross product of a vector with itself

$$\vec{a} \times \vec{a} = \vec{0}$$

Cross product of **parallel** vectors

$$\vec{a} \times \vec{b} = \vec{0} \quad \text{if } \vec{a} \parallel \vec{b}$$

Cross product with vectors scaled by s_1 and s_2

$$s_1 \vec{a} \times s_2 \vec{b} = s_1 s_2 (\vec{a} \times \vec{b})$$

Cross products are **not** commutative

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \quad (6)$$

Distributive law

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Distributive law

$$(\vec{a} + \vec{b}) \times (\vec{c} + \vec{d}) = \vec{a} \times \vec{c} + \vec{a} \times \vec{d} + \vec{b} \times \vec{c} + \vec{b} \times \vec{d}$$

Cross products are **not** associative

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

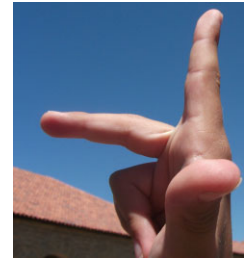
Vector triple cross product

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b}) \quad (7)$$

When \vec{b} is a unit vector

$$|\vec{a} \times \hat{b}|^2 = \vec{a} \cdot \vec{a} - (\vec{a} \cdot \hat{b})^2$$

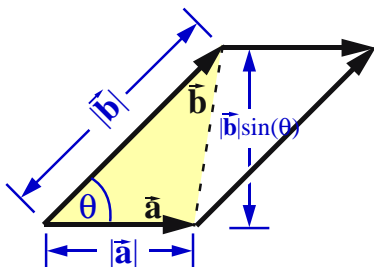
A mnemonic for $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b})$ is "**back cab**" - as in were you born in the **back** of a **cab**? Many proofs of this formula resolve \vec{a} , \vec{b} , and \vec{c} into orthogonal unit vectors (e.g., \hat{n}_x , \hat{n}_y , \hat{n}_z) and equate components.



2.10.2 Uses for the cross-product (\times)

Several uses for the cross-product in geometry, statics, and motion analysis, include calculating:

- **Perpendicular** vectors, e.g., $\vec{a} \times \vec{b}$ is perpendicular to both \vec{a} and \vec{b}
- **Moment** of a force or translational momentum, e.g., $\vec{r} \times \vec{F}$ and $\vec{r} \times m \vec{v}$
- **Velocity/acceleration** formulas, e.g., $\vec{v} = \vec{\omega} \times \vec{r}$ and $\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$
- **Area of a triangle** whose sides have length $|\vec{a}|$ and $|\vec{b}|$



The **area of a triangle** Δ is half the area of a parallelogram.^{a b}

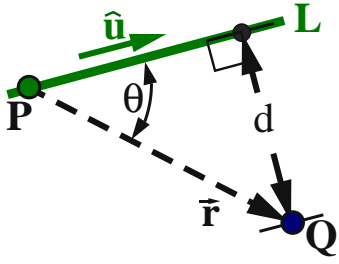
A geometrical interpretation of $|\vec{a} \times \vec{b}|$ is the **area of a parallelogram** having sides of length $|\vec{a}|$ and $|\vec{b}|$, hence

$$\Delta(\vec{a}, \vec{b}) = \frac{1}{2} |\vec{a} \times \vec{b}| \quad (8)$$

^aHomework 2.17 shows the utility of equation (8) for **surveying**.

^bSection 3.3 shows the utility of a cross-product for area calculations.

- **Distance** d between a line L and a point Q .

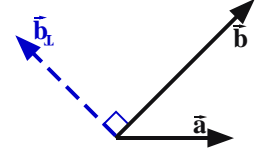


The line L (shown left) passes through point P and is parallel to the unit vector $\hat{\mathbf{u}}$. The **distance** d between line L and a point Q can be calculated as

$$d = |\vec{\mathbf{r}}^{Q/P} \times \hat{\mathbf{u}}| \stackrel{(5)}{=} |\vec{\mathbf{r}}^{Q/P}| \sin(\theta) \quad (9)$$

Note: See example in Hw 1.26. Other distance calculations are in Sections 3.2 and 3.3.

- The vector $\vec{\mathbf{b}}_{\perp}$ (shown right) is perpendicular to $\vec{\mathbf{b}}$ and is in the plane containing both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. It is calculated with the **vector triple cross product**:

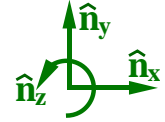


$$\vec{\mathbf{b}}_{\perp} = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{b}}$$

In general, $|\vec{\mathbf{b}}_{\perp}| \neq |\vec{\mathbf{b}}|$ and $\vec{\mathbf{b}}_{\perp}$ is not perpendicular to $\vec{\mathbf{a}}$.

2.10.3 Special case: Cross-products with right-handed, orthogonal, unit vectors

When $\hat{\mathbf{n}}_x, \hat{\mathbf{n}}_y, \hat{\mathbf{n}}_z$ are **orthogonal unit** vectors, it can be shown (see Homework 2.13) that the cross product of $\vec{\mathbf{a}} = a_x \hat{\mathbf{n}}_x + a_y \hat{\mathbf{n}}_y + a_z \hat{\mathbf{n}}_z$ with $\vec{\mathbf{b}} = b_x \hat{\mathbf{n}}_x + b_y \hat{\mathbf{n}}_y + b_z \hat{\mathbf{n}}_z$ happens to be equal to the **determinant** of the following matrix.



$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \det \begin{bmatrix} \hat{\mathbf{n}}_x & \hat{\mathbf{n}}_y & \hat{\mathbf{n}}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = (a_y b_z - a_z b_y) \hat{\mathbf{n}}_x - (a_x b_z - a_z b_x) \hat{\mathbf{n}}_y + (a_x b_y - a_y b_x) \hat{\mathbf{n}}_z$$

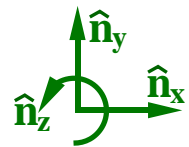
Optional: Special case of cross product as skew-symmetric matrix multiplication

Another way to calculate $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is with the following **skew symmetric matrix** multiplication. Homework 2.14 discusses the **inefficiencies** of calculating cross products with skew-symmetric matrix multiplication.

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \hat{\mathbf{n}}_{xyz}$$

2.10.4 Examples: Vector cross-products (\times)

The following shows how to use cross-products with the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, each which is expressed in terms of the orthogonal unit vectors $\hat{\mathbf{n}}_x, \hat{\mathbf{n}}_y, \hat{\mathbf{n}}_z$ shown to the right.



$$\vec{\mathbf{v}} = 7 \hat{\mathbf{n}}_x + 5 \hat{\mathbf{n}}_y + 4 \hat{\mathbf{n}}_z$$

$$\vec{\mathbf{w}} = 2 \hat{\mathbf{n}}_x + 3 \hat{\mathbf{n}}_y + 2 \hat{\mathbf{n}}_z$$

$$\vec{\mathbf{v}} \times \vec{\mathbf{w}} = \det \begin{bmatrix} \hat{\mathbf{n}}_x & \hat{\mathbf{n}}_y & \hat{\mathbf{n}}_z \\ 7 & 5 & 4 \\ 2 & 3 & 2 \end{bmatrix} = -2 \hat{\mathbf{n}}_x - 6 \hat{\mathbf{n}}_y + 11 \hat{\mathbf{n}}_z$$

$$\text{Area from vectors } \vec{\mathbf{v}} \text{ and } \vec{\mathbf{w}}: \Delta(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = \frac{1}{2} |\vec{\mathbf{v}} \times \vec{\mathbf{w}}| = \frac{1}{2} \sqrt{-2^2 + -6^2 + 11^2} = \frac{\sqrt{161}}{2} \approx 6.344$$

$$\vec{\mathbf{v}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \det \begin{bmatrix} \hat{\mathbf{n}}_x & \hat{\mathbf{n}}_y & \hat{\mathbf{n}}_z \\ 7 & 5 & 4 \\ -2 & -6 & 11 \end{bmatrix} = 79 \hat{\mathbf{n}}_x - 85 \hat{\mathbf{n}}_y - 32 \hat{\mathbf{n}}_z$$

2.11 Scalar triple product ($\cdot \times$ or $\times \cdot$)

The *scalar triple product* of vectors \vec{a} , \vec{b} , \vec{c} is the scalar defined in the various ways shown in equation (10). Homework 2.16 shows how *determinants* can calculate scalar triple products.

$$\text{ScalarTripleProduct} \triangleq \boxed{\vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \cdot \vec{c}} = \vec{b} \cdot \vec{c} \times \vec{a} = \vec{b} \times \vec{c} \cdot \vec{a} \quad (10)$$

Although parentheses make equation (10) clearer, i.e., $\text{ScalarTripleProduct} \triangleq \vec{a} \cdot (\vec{b} \times \vec{c})$, the parentheses are unnecessary because the cross product $\vec{b} \times \vec{c}$ **must** be performed before the dot product for a sensible result to be produced.

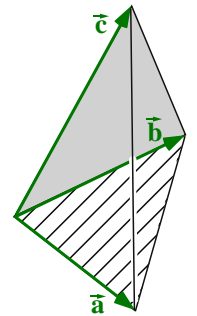
2.11.1 Scalar triple product and the volume of a tetrahedron

A geometrical interpretation of $\vec{a} \cdot \vec{b} \times \vec{c}$ is the *volume of a parallelepiped* having sides of length $|\vec{a}|$, $|\vec{b}|$, and $|\vec{c}|$. The formula for the *volume of a tetrahedron* whose sides are described by the vectors \vec{a} , \vec{b} , \vec{c} is



$$\text{Tetrahedron Volume} = \frac{1}{6} \vec{a} \cdot \vec{b} \times \vec{c}$$

This formula is used for volume calculations (e.g., highway *surveying* cut and fill), 3D *CAD*, solid geometry, and mass property calculations.

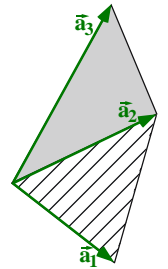


2.11.2 ($\times \cdot$) to change vector equations to scalar equations (see Hw 1.31)

Section 2.9.5 showed one method to form scalar equations from the vector equation $\vec{v} = \vec{0}$. A 2nd method expresses \vec{v} in terms of three non-coplanar (but not necessarily orthogonal or unit) vectors \vec{a}_1 , \vec{a}_2 , \vec{a}_3 , and writes the equally valid (but generally different) set of linearly independent scalar equations shown below.

$$\text{Method 2: if } \vec{v} = v_1 \vec{a}_1 + v_2 \vec{a}_2 + v_3 \vec{a}_3 = \vec{0} \Rightarrow \boxed{v_1 = 0 \quad v_2 = 0 \quad v_3 = 0}$$

Note: The proof that $v_i = 0$ ($i=1,2,3$) follows directly by substituting $\vec{v} = \vec{0}$ into equation (4.2).



Vectors are used with *surveying data* for volume cut-and-fill dirt calculations for highway construction