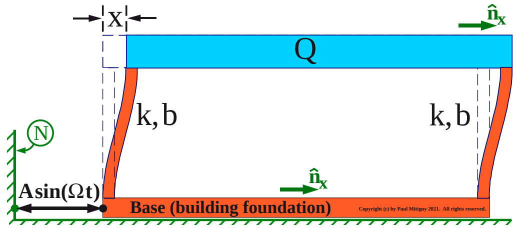
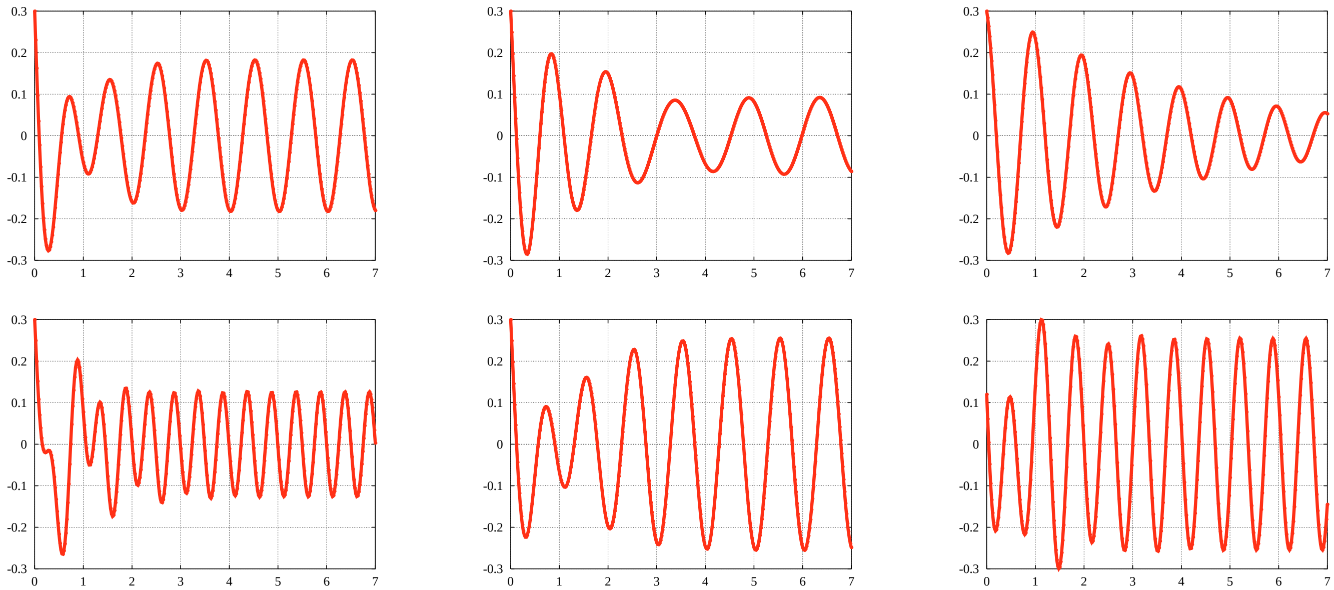
6.7 ♣ Dynamic response of a building in an earthquake (Section 13.2).

The base of a building vibrates because of an earthquake. The horizontal base motion is modeled as $A \sin(\Omega t)$ where $A = 0.1$ is the magnitude of the ground motion and Ω is the earthquake's frequency. The ODE governing the horizontal displacement x of the building's roof is



$$\ddot{x} \; + \; 2.4\,\dot{x} \; + \; 36\,x \; = \; 0.1\,\Omega^2 \, \sin(\Omega t)$$

Circle the graph of $x(t)$ that corresponds to $\Omega = 1 \text{ Hz} = 2\pi$ rad/sec.

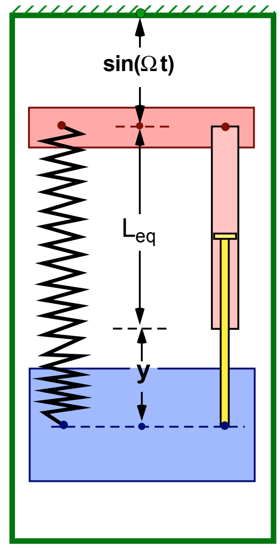


Explain: The steady state response has magnitude $|x_{ss}(t)| =$ ______ and a period of ___ sec.

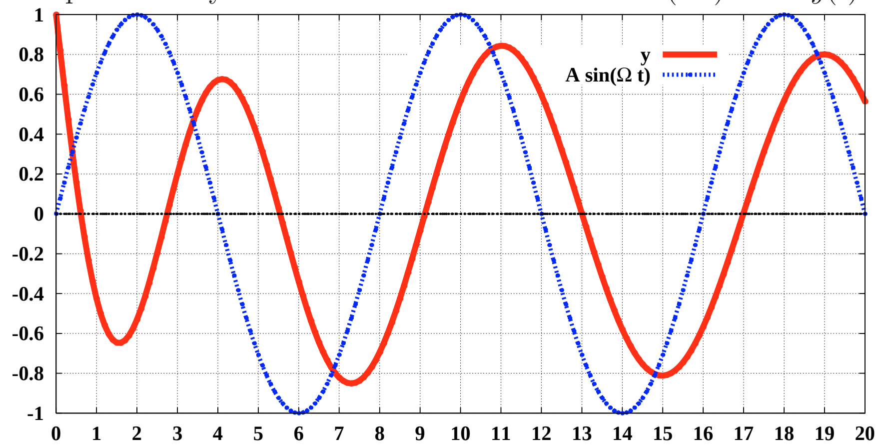
6.8 Determining ζ and ω_n from steady-state response. (Section 13.2).

Underdamped harmonically-forced vibrations for the system to the right are governed by the following 2^{nd}-order linear constant-coefficient ODE.

$$\ddot{y} \;+\; 2\,\zeta\,\omega_n\,\dot{y} \;+\; \omega_n^2\,y \;=\; A\,\Omega^2\,\sin(\Omega\,t)$$



Experimentally measured time-histories for $A\,\sin(\Omega\,t)$ and $y(t)$ are:



(a) Write the steady-state response $y(t)_{ss}$ in terms of the constants $B \triangleq |y_{ss}(t)|$, $\phi \triangleq \angle(y_{ss})$, etc. Complete the following table with numerical values for A, B, Ω, ϕ, etc.

Result: $y_{ss}(t) \;=\; \_\_\_\;\sin(\_\_\_\,t + \_\_\_)$

Description	Value	Units
Magnitude of function $A\,\sin(\Omega\,t)$	$A =$	
Magnitude of steady-state response $\lvert y_{ss}(t)\rvert$	$B =$	meter
Period of steady-state response		sec
Frequency of forcing function	$\Omega =$	rad/sec
Frequency of steady-state response		rad/sec
Phase of steady-state response with forcing function	$\phi =$	radians

Consider the phase ϕ the steady state response $y_{ss}(t)$ makes with the forcing function.

Negative phase means $y_{ss}(t)$	**lags/leads**	the forcing function, i.e., $y_{ss}(t)$ is **later/earlier**.
Positive phase means $y_{ss}(t)$	**lags/leads**	the forcing function, i.e., $y_{ss}(t)$ is **later/earlier**.
Negative phase shifts $y_{ss}(t)$	**left/right**	from the forcing function.
Positive phase shifts $y_{ss}(t)$	**left/right**	from the forcing function.

(b) Provide two algebraic equations that can be used to determine numerical values for ζ and ω_n. Next, classify those two equations by picking the relevant qualifiers from the following list. Lastly, describe a process for solving the previous two equations.

Result:

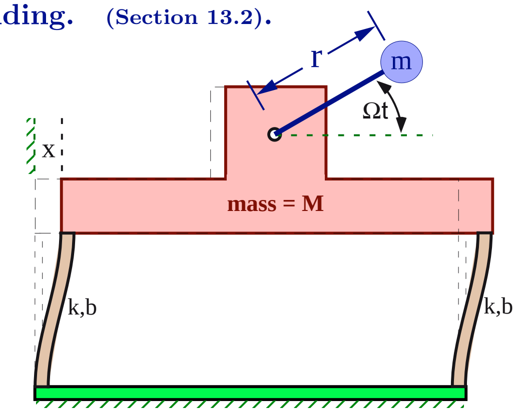
$\_\_\_ = \dfrac{\_\_\_}{\_\_\_}$ $\qquad$ $\_\_\_ = \_\_\_$

Uncoupled	**Linear**	**Homogeneous**	**Algebraic**
Coupled	**Nonlinear**	**Inhomogeneous**	**Differential**

Process:

6.9 † Dynamic response for an air-conditioner on a building. (Section 13.2).

An air conditioner is bolted to the roof of a one story building. The air conditioner's motor is unbalanced and its eccentricity is modeled as a particle of mass m attached to the distal end of a rigid rod of length r.



When the motor spins with angular speed Ω, it causes the building's roof of mass M to vibrate. The stiffness and material damping in each column that supports the roof is modeled as a linear horizontal spring (k) and linear horizontal damper (b). The ODE governing the horizontal displacement $x(t)$ of the building's roof is

$$\vec{\mathbf{F}} = m\,\vec{\mathbf{a}} \qquad \Rightarrow \qquad (M+m)\,\ddot{x} \;+\; 2\,b\,\dot{x} \;+\; 2\,k\,x \;=\; m\,r\,\Omega^2\,\sin(\Omega\,t)$$

$$\text{Alternate form for math:} \qquad \ddot{x} \;+\; 2\,\zeta\,\omega_n\,\dot{x} \;+\; \omega_n^2\,x \;=\; A\Omega^2\sin(\Omega\,t)$$

(a) Comparing the two previous equations, express $\omega_n,\ \zeta,\ A$ in terms of $M,\ m,\ b,\ k,\ r,\ t$.
Result:

$\omega_n = \square \qquad \zeta = \square \qquad A = \square$

(b) † The building's roof shakes too much. Comment on how $M,\ m,\ b,\ k,\ r$ affect $\zeta,\ \frac{\Omega}{\omega_n},\ |x_{ss}(t)|$. Complete **only** the blanks in the table (skip given answers) by writing $-$ (decreases), **0** (no effect), $+$ (increases), or ? (if it may decrease **or** increase). For the 2^{nd}-to-last column, assume the air conditioner's normal operating speed is $\Omega = 25\ \frac{\text{rad}}{\text{sec}}$ whereas for the last column, use $\Omega = 35\ \frac{\text{rad}}{\text{sec}}$.

Guess with intuition and verify with mathematics in Section 13.2.

Note: For unmodified building $\omega_n = 30\ \frac{\text{rad}}{\text{sec}}$	ζ	$\frac{\Omega}{\omega_n}$	$\Omega \approx 25\ \frac{\text{rad}}{\text{sec}}$ $\lvert x_{ss}(t)\rvert$	$\Omega \approx 35\ \frac{\text{rad}}{\text{sec}}$ $\lvert x_{ss}(t)\rvert$
Balancing the motor ($r \to 0$)	**0**	**0**	$-$	
Increasing the motor speed Ω (slightly)				
Decreasing the motor speed Ω (slightly)				
Adding mass to the roof (increasing M)			$+?$	$-$
Removing mass from the roof (decreasing M)			$-?$	?
Stiffening the support columns (increasing k)				
Adding damping to the columns (increasing b)				

You do not have to verify the $+?$ (mostly increase, but complicated) or $-?$ (mostly decrease, but complicated) answers.

(c) List two ways to change the **motor** and minimize the roof shaking.
Result: 1. ______. 2. ______

(d) For certain values of M, m, b, k, and r, this ODE simplifies to

$$\ddot{x} \;+\; 3\,\dot{x} \;+\; 900\,x \;=\; 1\,\text{x}\,10^{-4}\ \Omega^2\sin(\Omega t)$$

Calculate numerical values for ω_n and ζ. Fill in numerical values for the $x_{ss}(t)$ expressions.

Result:

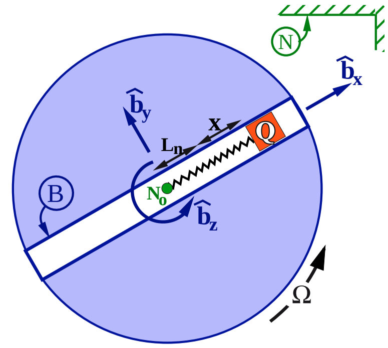
$\omega_n = \square\ \frac{\text{rad}}{\text{sec}}$

$\zeta = \square$

$\Omega\ \left(\frac{\text{rad}}{\text{sec}}\right)$	Steady-state part of $x(t)$
20	$x_{ss}(t) = 7.94\,\text{x}\,10^{-5} * \sin(\square\, t + -6.843^\circ)$
30	$x_{ss}(t) = \square * \sin(\square\, t + \square^\circ)$
40	$x_{ss}(t) = \square * \sin(\square\, t + \square^\circ)$

6.10 ♣ † Dynamic response of a particle in a horizontal spinning slot.

The figure to the right shows the top-view of a rigid body B consisting of a straight track welded to a circular disk that has a simple angular velocity on Earth (reference frame N). B's mass center B_{cm} is coincident with N_o, a point fixed in N. A particle Q slides with linear viscous damping along the track. A linear-spring connects Q to B_{cm}.



Top view

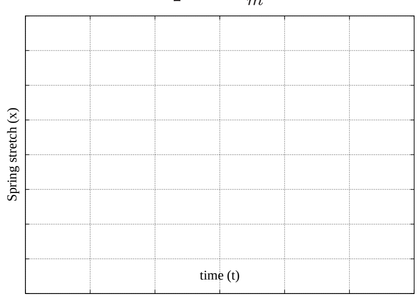
Quantity	Symbol	Type
Mass of Q	m	Constant
Linear viscous damping constant	b	Constant
Linear spring constant (translation)	k	Constant
Natural length of spring	L_n	Constant
Stretch of spring	x	Variable
$\hat{\mathbf{b}}_z$ measure of B's angular velocity in N	Ω	**Constant**

Using $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$, the ODE governing $x(t)$ is:
Alternate form for math:

$$\ddot{x} \;+\; \tfrac{b}{m}\,\dot{x} \;+\; (\tfrac{k}{m} - \Omega^2)\,x \;=\; \Omega^2\,L_n$$
$$\ddot{x} \;+\; 2\,\zeta\,\omega_n\,\dot{x} \;+\; \omega_n^2\;x \;=\; f(t)$$

Knowing b is smaller than m and k, Make a **rough sketch** of $x(t)$ for each case shown below. Use initial values of $x(0) = 0$ and $\dot{x}(0) = 0$. Explain each sketch in physics terms (force, mass, acceleration, initial condition) and mathematical terms (e.g., solution to ODE, ζ, ω_n, settling, stability, initial values, etc.).
Note: Determine the analytical solution $x(t)$ for the Medium Spin case.

Slow spin: $\frac{k}{m} \;>\; \Omega^2$



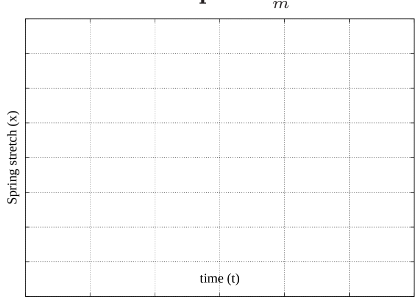
Physics explanation when x is positive:
The spring force can pull Q inward **less/more** (circle one) than centrifugal force can push Q outward from the disk's center.

Math explanation:
Solution is stable since $\omega_n^2 \;=\; (\dfrac{k}{m} - \Omega^2)$ is **negative/zero/positive**.

Solution for $x(t)$ settles to the positive number $x = \dfrac{\_\_\_\_}{\_\_\_\_}$.

If $b < m$, $b < k$, and $\frac{k}{m} \gg \Omega^2$, $x(t)$ is ____-damped since ____ $< \zeta <$ ____.

Medium spin: $\frac{k}{m} \;=\; \Omega^2$



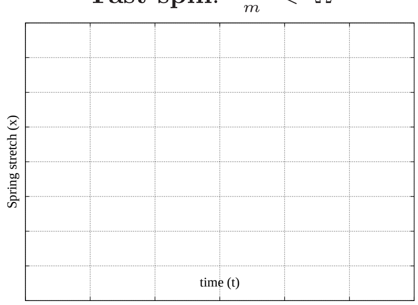
Physics explanation when x is positive:
The difference between the centrifugal force pushing Q outward and the spring force pulling Q inward is a force equal to ____.
After Q's speed increases enough, this centrifugal/spring force difference is matched by an inward ____ force equal to ____.

Math explanation:
Damping affects the particle's steady-state speed. **True/False**.
Natural frequency ω_n is **negative/zero/positive/imaginary**.

Solution for $\dot{x}(t)$ settles to the positive number $\dot{x} = \dfrac{\_\_\_\_}{\_\_\_\_}$.

$$x(t) \;=\; \_\_\_\_\;t \;+\; \frac{m^2\,\Omega^2\,L_n}{b^2}\,(e^{\_\_\_\_} - 1)$$

Fast spin: $\frac{k}{m} \;<\; \Omega^2$



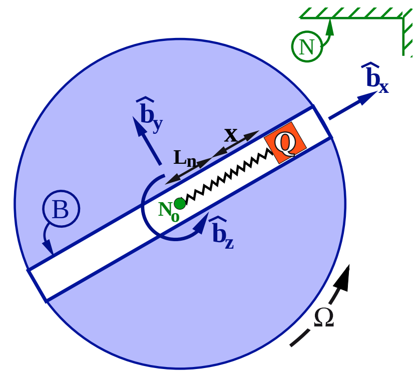
Physics explanation when x is positive:
Centrifugal force pushes Q outward **less/more** than the spring force pulls Q inward, and the force difference **decreases/increases** with x.

Math explanation:
Damping affects the solution's exponential growth rate. **True/False**.
Natural frequency ω_n is **negative/zero/positive/imaginary**.
Solution for $x(t)$ is unstable and grows exponentially. **True/False**.

6.11 ♣ Dynamic response for various Ω for particle in horizontal spinning slot.

The following figure shows the **top-view** of a rigid body B consisting of a straight track welded to a **horizontal** circular disk that has a simple angular velocity on Earth (reference frame N). A particle Q slides along the greased track. A linear-spring connects Q to a point N_o that is fixed in N and coincident with B's center.

Quantity	Symbol	Type
Mass of Q	m	Constant
Viscous damping constant	b	Constant
Linear spring constant	k	Constant
Natural length of spring	L_n	Constant
Stretch of spring	x	Variable
Measure of B's angular velocity in N	$\Omega(t)$	**Specified**



Top view

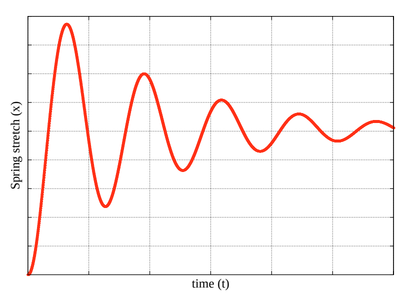
ODE for $x(t)$: $$m\,\ddot{x} \;+\; b\,\dot{x} \;+\; (k - m\,\Omega^2)\,x \;=\; m\,\Omega^2\,L_n$$

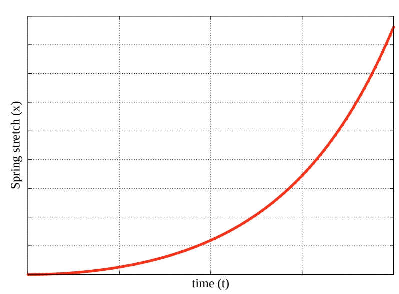
Alternate form: $$\ddot{x} \;+\; 2\,\zeta\,\omega_n\,\dot{x} \;+\; \omega_n^2\;x \;=\; f(t)$$

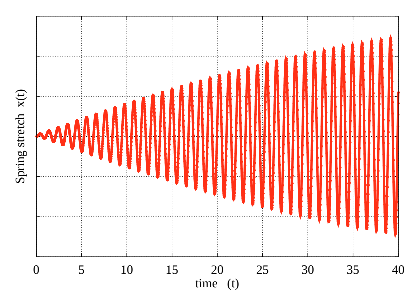
Q's motion depends on whether Ω is a small constant, large constant, or harmonically forced at resonance. Label each graph with one of **A**, **B**, **C**, **D**, **E**, or **F** as described below.

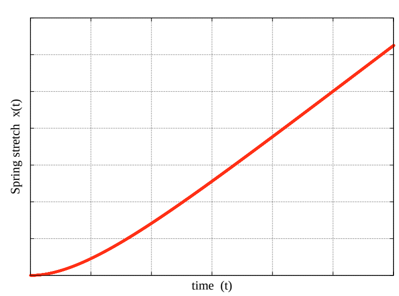
A. Strong spring	$k > m\,\Omega^2$	**with slow**	**constant spin** Ω	(small centrifuge)
B. Medium spring	$k = m\,\Omega^2$	**with medium**	**constant spin** Ω	(medium centrifuge)
C. Weak spring	$k < m\,\Omega^2$	**with fast**	**constant spin** Ω	(large centrifuge)
D. Strong spring	$k > m\,\Omega^2$	**harmonic forcing near resonance***		$\Omega = \cos(\frac{1}{2}\sqrt{\frac{k}{m}}\,t)$
E. Very strong spring	$k \gg m\,\Omega^2$	**harmonic forcing near resonance***		$\Omega = \cos(\frac{1}{2}\sqrt{\frac{k}{m}}\,t)$
F. Strong spring	$k > m\,\Omega^2$	**harmonic forcing above resonance***		$\Omega = \cos(2\sqrt{\frac{k}{m}}\,t)$

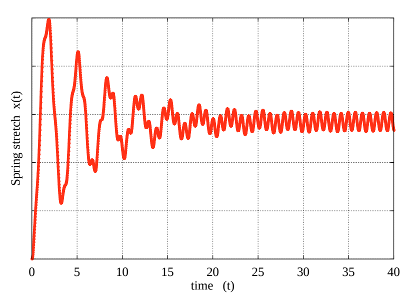
*Note: With harmonic forcing $\Omega = \cos(\sqrt{\frac{k}{m}}\,t)$, $\Omega^2 \le 1$ since the cosine function returns values between −1 and 1.

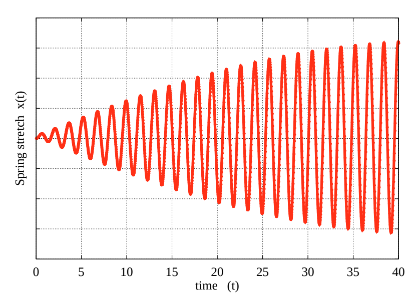










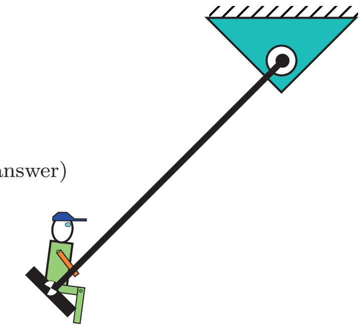


6.12 ♣ Power/energy-rate principle concepts. (Chapter 9).

A force of 20 Newtons is to be briefly applied to a child on a swing (modeled as a particle on a 2 m rope). Determine the optimal time to push the child to change the swinging child's kinetic energy.
The duration of this force is short compared to the swing's period of oscillation ($\tau_{\text{period}} \approx \frac{2\,\pi}{\sqrt{g/L}} \approx \frac{2\,\pi}{\sqrt{9.8/2}} \approx 2.8$ sec).

To best ***increase*** kinetic energy, push the child ***forward*** when (circle the best answer)

- The child just starts moving forward at the top of the swing
- The child is moving quickly forward at the bottom of the swing
- The child is moving quickly backward at the bottom of the swing
- The child just stops moving backward at the top of the swing
- Other (**explain**):



To best ***decrease*** kinetic energy, push the child ***forward*** when (circle the best answer)

- The child just starts moving forward at the top of the swing
- The child is moving quickly forward at the bottom of the swing
- The child is moving quickly backward at the bottom of the swing
- The child just stops moving backward at the top of the swing
- Other (**explain**):

Reason: Putting ______ in the direction of ______ increases energy most efficiently.

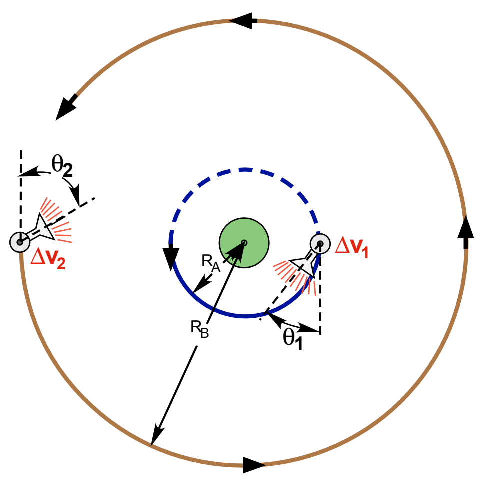
6.13 ♣ Power/energy-rate principle: Minimum fuel-use orbit transfer. (Chapter 9).

To thrust a satellite from low circular orbit about Earth to a higher circular orbit, an impulse is provided at two instants.

The first impulse can be directed radially outward, tangent to the satellite's circular orbit, or directed at some angle θ_1 from the satellite's orbital tangent. The first impulse puts the satellite into an elliptical orbit.

The second impulse is applied at apogee (when the satellite is furthest from Earth) and is directed at an angle θ_2 from the orbital tangent. The second impulse changes the orbit from elliptical to circular.

Using engineering insights, provide values for θ_1 and θ_2 that minimize the fuel required for this orbit transfer, a reason for choosing these values, and **roughly sketch** the trajectory.



Result: $\theta_1 = \_\_\,^\circ$ $\theta_2 = \_\_\,^\circ$

Reason: Putting ______ in the direction of ______ increases energy most efficiently.

Note: In 1925, Walter Hohmann described a minimum-fuel orbital maneuver (***Hohmann transfer orbit***) that uses two engine impulses to move a spacecraft between two coplanar circular orbits.

Note: See the related, similar question in Chapter 9.