Geometry and calculus – circles, triangles, derivatives and integrals.

Show work – except for ♣ fill-in-blanks-problems (print .pdf from www.MotionGenesis.com ⇒ Textbooks ⇒ Resources).

1.1 ♣ Solving problems – what engineers do.

Understanding this material results from **doing** problems. Many problems are guided to help you synthesize processes (imitation). You are encouraged to work by yourself or with colleagues/instructors and use the textbook's reference theory and other resources.

Confucius 500 B.C.
"I hear and I forget.
I see and I remember.
I $\boxed{\quad}$ and I understand."

"By three methods we may learn wisdom:
First, by reflection, which is noblest;
Second, by imitation, which is easiest;
Third by experience, which is the bitterest."

1.2 ♣ PEMDAS (Parentheses, Exponents, Multiplication/Division, Addition/Subtraction).

$9 - 4 \;*\; 2 \;=\; \boxed{\quad}$

$\dfrac{4 \;*\; 5+4}{2} + 5 \;=\; \boxed{\quad}$

$36 \,/\; 3*3 \;-\; 12 \;=\; \boxed{\quad}$

$2*5^{\;2} \;-\; 25 \;=\; \boxed{\quad}$

$$\left[+\sqrt{(3^3 \;+\; 23) \,*\, \tfrac{1}{2}} \;*\; 2 \;+\; 2+3 \div 3\right] * (5+6) \;=\; \boxed{\quad}$$

$-3^2 \;=\; \boxed{\quad}$

$2^{3^2} \;=\; \boxed{\quad}$

1.3 ♣ Unit conversions between U.S. and SI (Standard International). (Guess and check Section 2.3).

Complete each blank with one of the following numbers: **0.45**, **1**, **2.54**, **32.2**.

Length	1 inch $\triangleq$ $\boxed{\quad}$ cm	**Mass**	1 lbm $\approx$ $\boxed{\quad}$ kg	1 slug $\approx$ $\boxed{\quad}$ lbm	
Force	1 Newton $\triangleq$ $\boxed{\quad}$ $\frac{\text{kg m}}{\text{s}^2}$		1 lbf $\triangleq$ $\boxed{\quad}$ $\frac{\text{slug ft}}{\text{s}^2}$	1 lbf $\approx$ $\boxed{\quad}$ $\frac{\text{lbm ft}}{\text{s}^2}$	

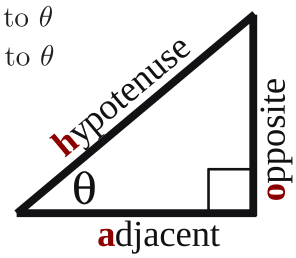
1.4 ♣ SohCahToa: Sine, cosine, tangent as ratios of sides of a right triangle. (Section 2.6)

The following shows a ***right triangle*** with one of its angles labeled as θ.
Write definitions for sine, cosine, and tangent in terms of:

- **h**ypotenuse – the triangle's longest side (opposite the 90° angle).
- **o**pposite – the side opposite to θ
- **a**djacent – the side adjacent to θ

Note: A mnemonic for these definitions is "***SohCahToa***".

Note: A ***right triangle*** is a triangle with a 90° angle.



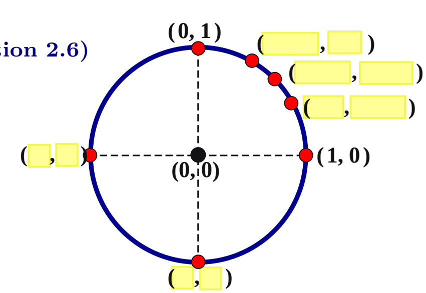
$$\sin(\theta) \triangleq \frac{\boxed{\quad}}{\boxed{\quad}}$$

$$\cos(\theta) \triangleq \frac{\boxed{\quad}}{\boxed{\quad}}$$

$$\tan(\theta) \triangleq \frac{\boxed{\quad}}{\boxed{\quad}} = \frac{\sin(\theta)}{\cos(\theta)}$$

1.5 ♣ Memorize common sines and cosine. (Section 2.6)

$\sin(0°) = \boxed{\quad}$	$\cos(0°) = \boxed{\quad}$
$\sin(30°) = \boxed{\quad}$	$\cos(30°) = \boxed{\quad}$
$\sin(45°) = \boxed{\quad}$	$\cos(45°) = \boxed{\quad}$
$\sin(60°) = \boxed{\quad}$	$\cos(60°) = \boxed{\quad}$
$\sin(90°) = \boxed{\quad}$	$\cos(90°) = \boxed{\quad}$



Label the coordinates of each point on the unit circle.

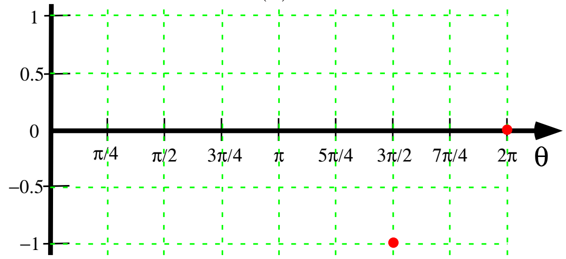
1.6 ♣ Graphing sine and cosine - (a now-obvious invention from 1730 A.D.) (Section 2.6.2)

Graph sine and cosine as functions of the angle θ in radians over the range $0 \leq \theta \leq 2\pi$.

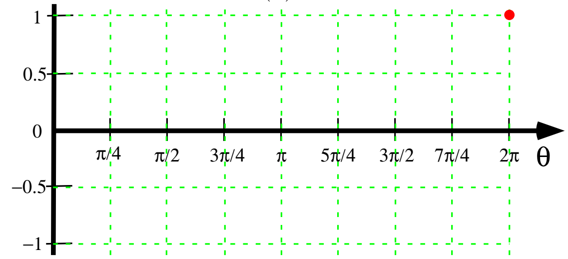
The mathematician was first to regard sine and cosine as **functions** (not just ratios of sides of a triangle).

Result:

$\sin(\theta)$ vs. θ



$\cos(\theta)$ vs. θ



1.7 ♣ Memorize sine and cosine addition formulas (Section 2.6.1).

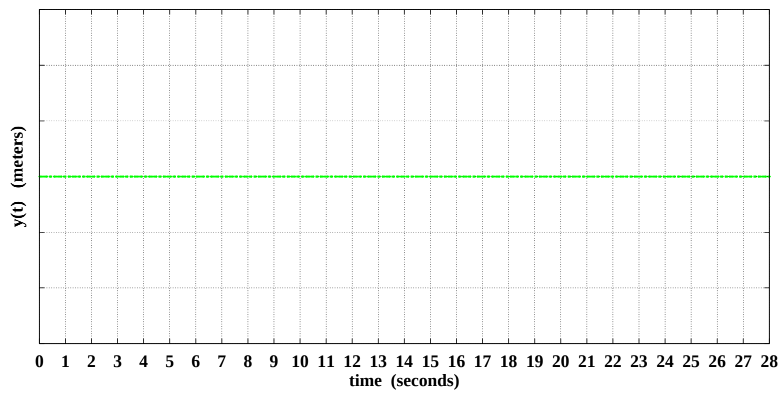
$$\sin(\alpha+\beta) \;=\; \sin(\alpha) \qquad + \qquad \text{\textit{Addition formula for sine}}$$
$$\cos(\alpha+\beta) \;=\; \cos(\alpha) \qquad \text{\textit{Addition formula for cosine}}$$

1.8 ♣ Graph sine functions and identify amplitude, frequency, and phase (Section 2.6.2).

For $0 \leq t \leq 28$ sec, graph the following functions (label your axes).

$$y_A(t) \;=\; 3 * \sin(\frac{\pi}{12}\,t)$$
$$y_B(t) \;=\; 3 * \sin(\frac{\pi}{12}\,t \;-\; \frac{\pi}{4})$$

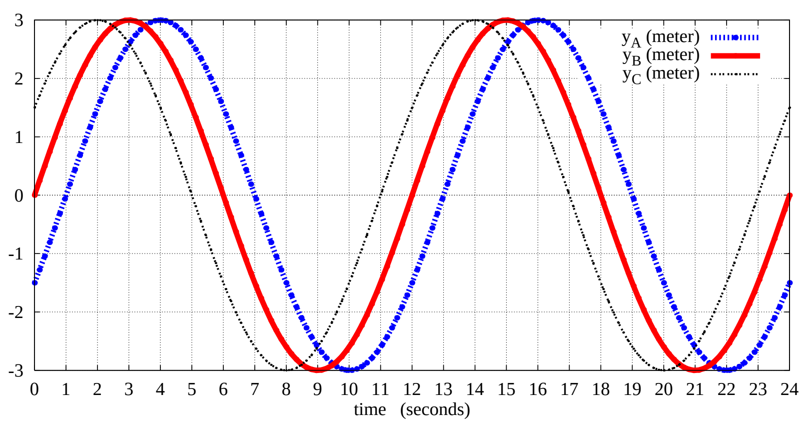
Since the ***phase*** of $y_B(t)$ is ,
$y_B(t)$ **leads/lags** $y_A(t)$.



In general **negative/positive** phase is **lag** (later), which shifts a curve **left/right**.
In general **negative/positive** phase is **lead** (earlier), which shifts a curve **left/right**.

1.9 ♣ Identifying amplitude, frequency, and phase for sine functions (Section 2.6.2).

Graphed below are the time-dependent functions $y_A(t)$, $y_B(t)$, $y_C(t)$. Determine numerical values and units for their non-negative ***amplitudes*** B, non-negative ***frequencies*** Ω, and ***phase*** ϕ ($-\pi < \phi_i \leq \pi$).



$y_i(t) = B\,\sin(\Omega\,t + \phi_i)$ $(i = A, B, C)$

	Value	Units
$B =$		
$\Omega =$		
$\phi_A =$		
$\phi_B =$		
$\phi_C =$		

1.10 ♣ Ranges for arguments and return values for inverse trigonometric functions.

Determine all real return values and argument values for the following **real** trigonometric and inverse-trigonometric functions in computer languages such as Java, C^{++}, **M**otion**G**enesis, and MATLAB®.

Possible return values	Function	Possible argument values	Note
$\square \le z \le \square$	$z = \cos(x)$	$\square < x < \square$	
$\square \le z \le \square$	$z = \sin(x)$	$\square < x < \square$	
$\boxed{-\infty} < z < \boxed{\infty}$	$z = \tan(x)$	$\boxed{-\infty} < x < \boxed{\infty}$	$x \ne \frac{-\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \ldots$
$\square \le z \le \square$	$z = \text{acos}(x)$	$\square \le x \le \square$	
$\square \le z \le \square$	$z = \text{asin}(x)$	$\square \le x \le \square$	
$\boxed{-\pi/2} < z < \boxed{\pi/2}$	$z = \text{atan}(x)$	$\boxed{-\infty} < x < \boxed{\infty}$	
$\square < z \le \square$	$z = \texttt{atan2}(y, x)$	$\square < y < \square$ $\square < x < \square$	$\texttt{atan2}(0,0)$ is undefined

1.11 ♣ Notations for derivatives. (Section 2.8.1).

Date	Person	Symbols for 1^{st}, 2^{nd}, and 3^{rd} derivatives		
1675	$\square$	$\frac{dy}{dt}$	$\frac{d^2y}{dt^2}$	$\frac{d^3y}{dt^3}$
1675	$\square$	$\dot{y}$	$\ddot{y}$	$\dddot{y}$
1797	$\square$ (trained by Euler)	y'	y''	y'''
1850	Cauchy/Weierstrauss	$\lim\limits_{h \to 0} \frac{y(t+h) - y(t)}{h}$	?	?
1786	Legendre (introduced partials then abandoned)	$\frac{\partial y}{\partial x}$	$\frac{\partial^2 y}{\partial x^2}$	$\frac{\partial^3 y}{\partial x^3}$
1841	Jacobi (re-introduced partials again)			

There was bitter rivalry between Newton and Leibniz, and the notations of Leibniz and Newton are not entangled.
For example, $\frac{d\dot{y}}{dt}$ is written in Leibniz's notation as $\square$ or Newton's as $\square$.

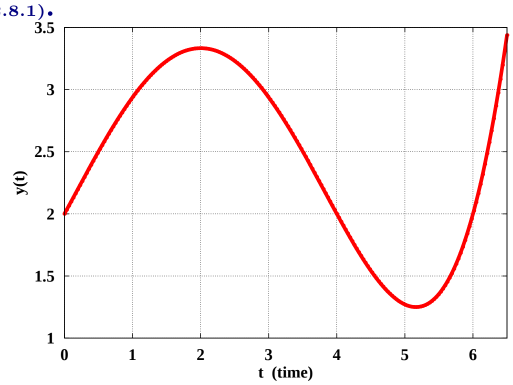
1.12 ♣ Leibniz's shorthand notation for 3^{rd} derivatives. (Section 2.8.1).

Write the explicit expression for the following 3^{rd} derivative (so it only contains 1^{st} derivatives).

Result:

$$\frac{d^3y}{dt^3} \triangleq \square$$

1.13 ♣ Geometric interpretation of a derivative. (Section 2.8.1).



Estimate the 1^{st}-derivative of the function $y(t)$ shown to the right at $t = 0,\ 2,\ 4,\ 6$.

Pick your answers from: **-1**, **0**, **1**, **2**.

Result: $\left.\frac{dy}{dt}\right|_{t=0} = \square$ $\quad \left.\frac{dy}{dt}\right|_{t=2} = \square$

$\left.\frac{dy}{dt}\right|_{t=4} = \square$ $\quad \left.\frac{dy}{dt}\right|_{t=6} = \square$

Estimate the **sign** of the 2^{nd}-derivative of $y(t)$ from the answers **−**, **0**, or **+**.
Answer **0** when the absolute value of the 2^{nd}-derivative is estimated to be less than 0.5.

Result: $\left.\frac{d^2y}{dt^2}\right|_{t=0}$ is $\square$ $\quad \left.\frac{d^2y}{dt^2}\right|_{t=2}$ is $\square$ $\quad \left.\frac{d^2y}{dt^2}\right|_{t=4}$ is $\square$ $\quad \left.\frac{d^2y}{dt^2}\right|_{t=6}$ is $\square$

1.14 ♣ Derivatives of commonly-encountered functions. (Section 2.8.5).

Differentiate the following functions that depend on t (time). Ensure answers involving x are valid when x is either constant or depends on time, e.g., when $x = t^3$.

Result:

$$\frac{d}{dt}\,t^2 = \boxed{\quad} \qquad \frac{d}{dt}\,t^3 = \boxed{\quad} \qquad \frac{d}{dt}\,t^{47} = \boxed{\quad}$$

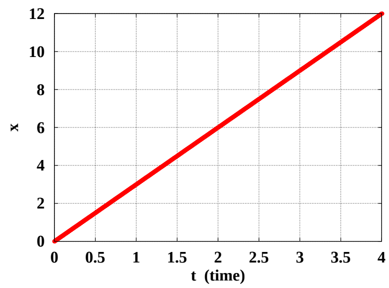
$$\frac{d}{dt}\,\sin(t) = \boxed{\quad} \qquad \frac{d}{dt}\,\cos(t) = \boxed{\quad} \qquad \frac{d}{dt}\,\cos(x) = \boxed{\quad}$$

$$\frac{d}{dt}\,e^t = \boxed{\quad} \qquad \frac{d}{dt}\,\ln(t) = \boxed{\quad} \qquad \frac{d}{dt}\,\ln(x) = \boxed{\quad}$$

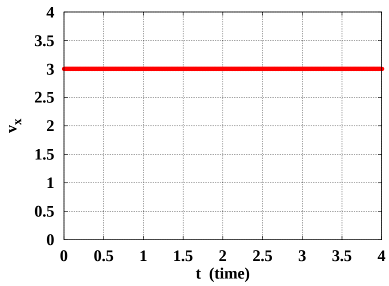
1.15 ♣ Geometric interpretations of a derivative. (Section 2.8.1).

Complete the missing analytical statements and graph the missing functions.

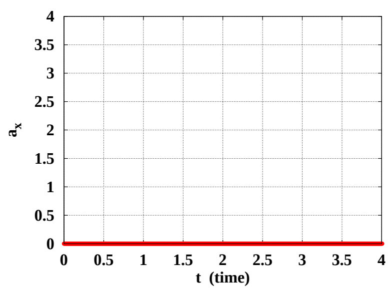
$x = 3\,t$



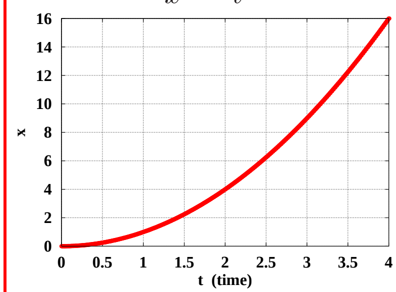
$v_x \triangleq \dfrac{dx}{dt} = \boxed{\quad}$



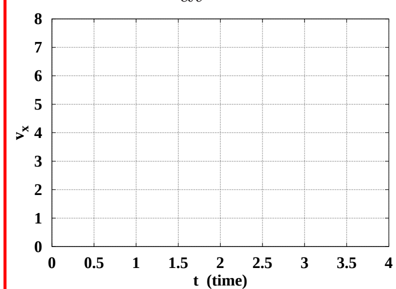
$a_x \triangleq \dfrac{d^2x}{dt^2} = \boxed{\quad}$



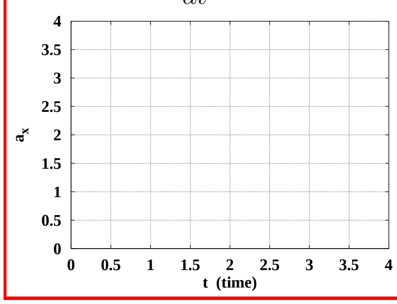
$x = t^2$



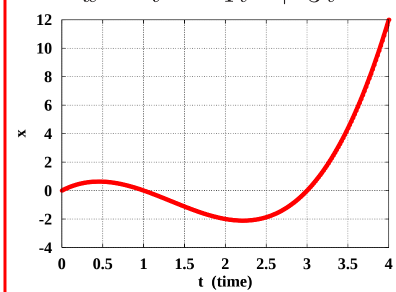
$v_x \triangleq \dfrac{dx}{dt} = \boxed{\quad}$



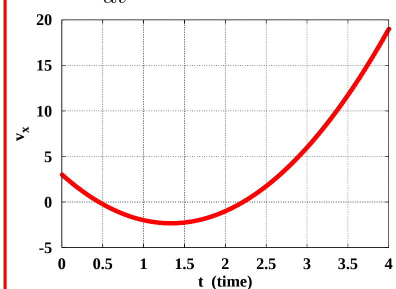
$a_x \triangleq \dfrac{d^2x}{dt^2} = \boxed{\quad}$



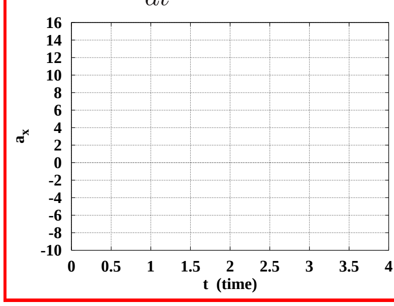
$x = t^3 - 4\,t^2 + 3\,t$



$v_x \triangleq \dfrac{dx}{dt} = \boxed{\quad}$



$a_x \triangleq \dfrac{d^2x}{dt^2} = \boxed{\quad}$



1.16 ♣ Good product rule for differentiation. **(Section 2.8.7).**

The ***good product rule for differentiation*** that works when u and v are scalars, vectors, or matrices is (circle the correct answer):

$$\frac{d(u*v)}{dt} = \frac{du}{dt} * v + u * \frac{dv}{dt} \qquad \frac{d(u*v)}{dt} = u * \frac{dv}{dt} + v * \frac{du}{dt} \qquad \frac{d(u*v)}{dt} = v * \frac{du}{dt} + u * \frac{dv}{dt}$$

1.17 Differentiating quotients: Use the product rule and exponents. **(Division - "Just say No").**

Although the "***quotient rule***" can be used to calculate the derivative with respect to t of the ratio of two functions $\frac{f(t)}{g(t)}$, it can be easier to rewrite the ratio as $f(t) * g(t)^{-1}$ then use the ***product rule***. Use this idea to first rewrite the following ratio of two functions as a product and then use the ***product rule*** to calculate its derivative.

Result: $\frac{\ln(t)}{t^2} = \boxed{}$ $\qquad \frac{d}{dt}\left[\ln(t)\,/\,t^2\right] = \boxed{}$

1.18 ♣ Example of the "good product rule" for differentiation. **(Takes less than 2 minutes).**

The "good" product rule is easy-to-use for ***very quickly*** differentiating complex expressions. Knowing x and y are variables that depend on the independent variable t (time), determine the ordinary time-derivative of the function f when[1]

$$f(t) \;=\; \sin(t) * \cos(x+y) * (\dot{x})^2 * e^t * \ln(y) \,/\, x$$

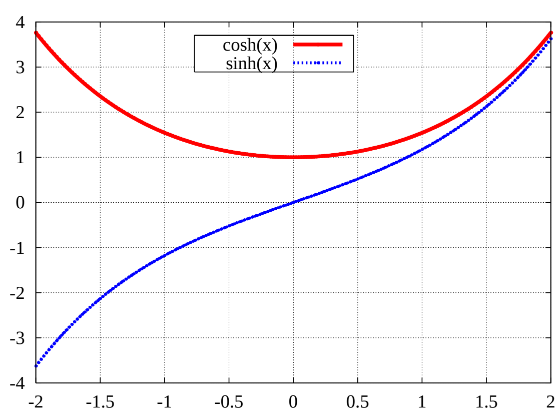
Result:

$$\begin{aligned}\frac{df}{dt} &= \cos(t) * \cos(x+y) * (\dot{x})^2 * e^t * \ln(y) \,/\, x \\ &- \sin(t) * \sin(x+y) * (\dot{x}+\dot{y}) * (\dot{x})^2 * e^t * \ln(y) \,/\, x \\ &+ \boxed{} \\ &+ \boxed{} \\ &+ \boxed{} \\ &- \boxed{}\end{aligned}$$

1.19 The amazing function e^x. Example: The hyperbolic cosine and sine functions.

The ***hyperbolic cosine*** and ***hyperbolic sine*** functions are defined below and plotted to the right.

$$\cosh(x) \triangleq \frac{e^x + e^{-x}}{2} \qquad \sinh(x) \triangleq \frac{e^x - e^{-x}}{2}$$



Show ***how*** to find their ordinary derivatives with respect to x in terms of hyperbolic functions.

Result:

$$\frac{d\,[\cosh(x)]}{dx} = \boxed{}$$

$$\frac{d\,[\sinh(x)]}{dx} = \boxed{}$$

[1]Symbols for the 1st and 2nd ordinary time-derivatives of x include $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ (introduced by ***Leibniz***), $\dot{x}$ and $\ddot{x}$ (introduced by ***Newton***), and `x'` and `x''` (introduced by ***Lagrange*** and used by **M**otion**G**enesis).

1.20 Differentiation concepts. (Section 2.8.10).

The following equation relates the dependent variable y to the independent variable t.

$$y^4 \;-\; 8\,y \;=\; 3\,t^2 \;+\; \sin(t)$$

Find a general expression for the ordinary derivative $\frac{dy}{dt}$ in terms of t and y.

Find a **numerical** value for $\frac{dy}{dt}$ at $t = 0$ when y is **positive**.

Hint: The value of y is not arbitrary. If you encounter difficulty, first do Homeworks 1.21 and 1.24.

Result:

$$\frac{dy}{dt} = \boxed{\quad\quad\quad} \qquad\qquad \left.\frac{dy}{dt}\right|_{t=0} = \boxed{\quad}$$

1.21 Differentiation concepts. (Section 2.8.10).

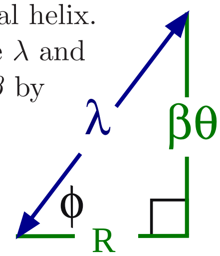
Calculate the ordinary time-derivative of $y = 5^t$.

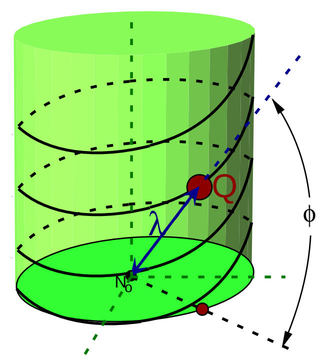
Result:

$$\frac{dy}{dt} \;=\; \boxed{\quad}\,5^t \;=\; \boxed{\quad}\,y$$

1.22 Review of explicit and implicit differentiation. (Section 2.8.10).

The figure to the right shows a point Q on a cylindrical helix. Two geometrically significant quantities are a distance λ and an angle ϕ that are related to two **constants** R and β by





$$\lambda^2 \;=\; R^2 \;+\; (\beta\,\theta)^2 \qquad\qquad \tan(\phi) \;=\; \frac{\beta\,\theta}{R}$$

Determine $\dot{\lambda}$ and $\dot{\phi}$ (time-derivatives of λ and ϕ) in terms of θ, $\dot{\theta}$, R, β, etc., using the two methods described below. Note: $\frac{\partial\,\text{atan}(x)}{\partial x} \;=\; \frac{1}{1+x^2}$

(a) ***Explicit differentiation***

Solve explicitly for λ and ϕ and then differentiate the resulting expression.

Result:

$$\lambda \;=\; \sqrt{R^2 \;+\; (\beta\,\theta)^2} \qquad\qquad \phi \;=\; \text{atan}(\tfrac{\beta\,\theta}{R})$$

$$\dot{\lambda} \;=\; \boxed{\quad\quad\quad} \qquad\qquad \dot{\phi} \;=\; \boxed{\quad\quad\quad}$$

(b) ***Implicit differentiation***

Differentiate the equations involving λ^2 and $\tan(\phi)$ and then solve for $\dot{\lambda}$ and $\dot{\phi}$.

Result:

$$\dot{\lambda} \;=\; \boxed{\quad\quad} \qquad\qquad \dot{\phi} \;=\; \boxed{\quad\quad\quad} \;=\; \tfrac{\beta\,R}{\lambda^2}\,\dot{\theta}$$

(c) **Explicit/Implicit** differentiation of λ is easier and computationally more efficient.

1.23 ♣ Review of partial and ordinary differentiation. (Section 2.8.2).

The kinetic energy K of the system to the right can be written in terms of constants m^A, m^Q, L and time-dependent variables x, θ, as

$$K \;=\; \frac{1}{2}\,m^A\,\dot{x}^2 \;+\; \frac{1}{2}\,m^Q\,[\dot{x}^2 + L^2\,\dot{\theta}^2 + 2\,L\,\cos(\theta)\,\dot{x}\,\dot{\theta}]$$

Use partial and ordinary differentiation to form the following ingredients for ***Lagrange's equations of motion***.

$$\frac{\partial K}{\partial \theta} = \boxed{\quad\quad\quad} \qquad \frac{\partial K}{\partial \dot{\theta}} = \boxed{\quad\quad\quad} \qquad \frac{d}{dt}\,\frac{\partial K}{\partial \dot{\theta}} = \boxed{\quad\quad\quad\quad\quad}$$

1.24 Differentiation concepts: What is dt? (Section 2.8.3).

A continuous function $z(t)$ depends on $x(t)$, $y(t)$, and time t as $\quad z = x + y^2 \sin(t)$

At a certain instant of time, $y = 1$ and z simplifies to $\quad z = x + \sin(t)$

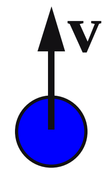
Find the time-derivative of z at the instant when $y = 1$.

Result: $\quad \left.\frac{dz}{dt}\right|_{y=1} =$

1.25 ♣ Differentiation concepts. (Section 2.8.3 and previous problem).

The scalar v measures a baseball's upward-velocity. Knowing $v = 0$ when the ball reaches maximum height near Earth ($g \approx 9.8 \,\frac{\text{m}}{\text{s}^2}$), decide if the following statement about v's time derivative is true.

$$\frac{dv}{dt} = \frac{d\,(0)}{dt} = 0 \qquad \textbf{True/False}$$



Explain:

1.26 ♣ Integrals of commonly-encountered functions. (Section 2.9).

Calculate the following indefinite integrals in terms of an indefinite constant C (regard t as positive).

Result:

$\int t^2\, dt =$ $\qquad \int t^3\, dt =$ $\qquad \int t^8\, dt =$

$\int t^{-3}\, dt =$ $\qquad \int t^{-2}\, dt =$ $\qquad \int t^{-1}\, dt =$

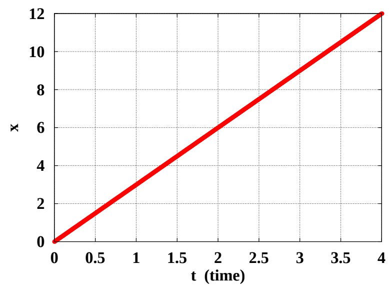
$\int \sin(t)\, dt =$ $\qquad \int \cos(t)\, dt =$ $\qquad \int e^t\, dt =$

$\int 5\, dt =$ $\qquad \int 5/t\, dt =$ $\qquad \int (5 + \frac{1}{t})\, dt =$

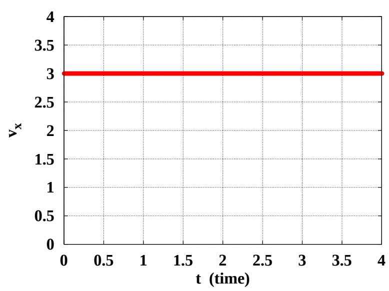
1.27 ♣ **Geometric interpretations of integrals.** **(Section 2.9).**

Complete the missing analytical statements and graph the missing functions.
Note: Constants of integration can be deduced from graphs.

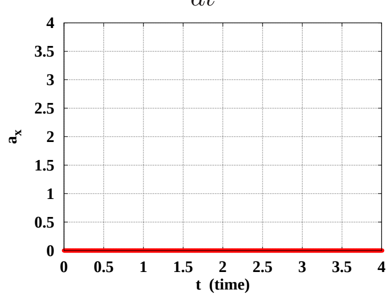
$x \; = \; \boxed{\quad}$



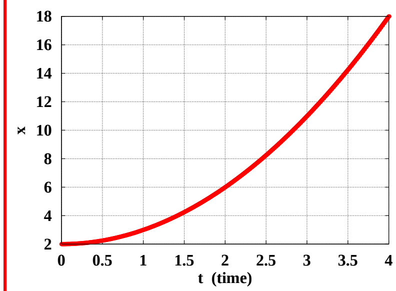
$v_x \; \triangleq \; \frac{dx}{dt} \; = \; \boxed{\quad}$



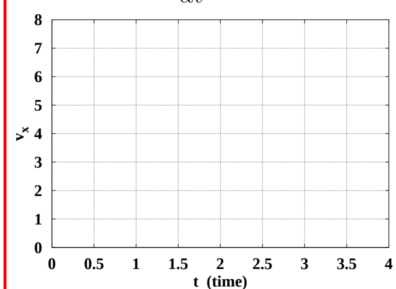
$a_x \; \triangleq \; \frac{d^2x}{dt^2} \; = \; 0$



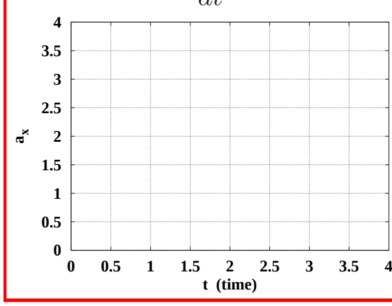
$x \; = \; \boxed{\quad}$



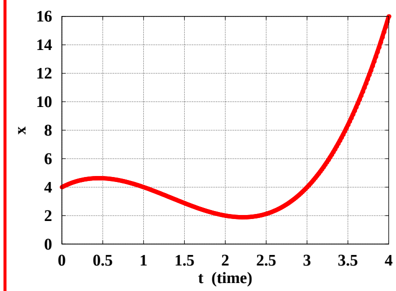
$v_x \; \triangleq \; \frac{dx}{dt} \; = \; \boxed{\quad}$



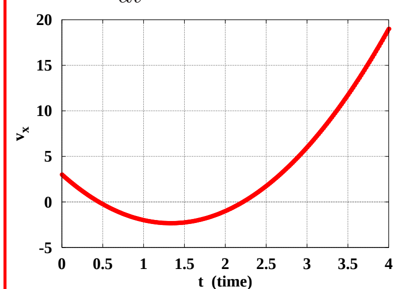
$a_x \; \triangleq \; \frac{d^2x}{dt^2} \; = \; 2$



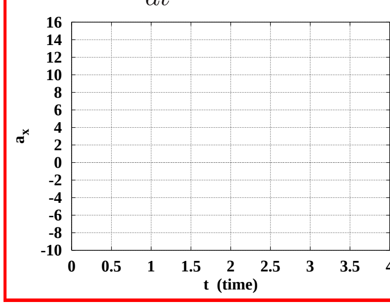
$x \; = \; \boxed{\quad}$

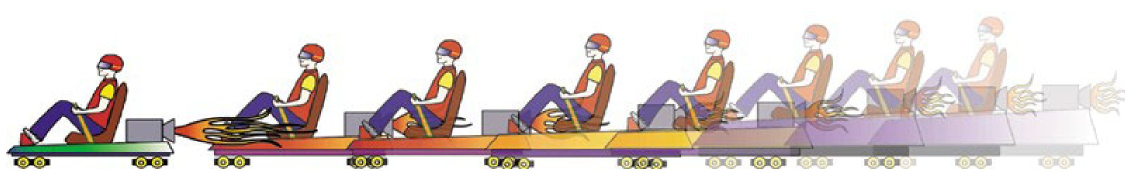


$v_x \; \triangleq \; \frac{dx}{dt} \; = \; 3\,t^2 \; - \; 8\,t \; + \; 3$



$a_x \; \triangleq \; \frac{d^2x}{dt^2} \; = \; \boxed{\quad}$





1.28 ♣ **Optional:** Math history (.pdf at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources).

Connect the mathematical event with its approximate date.

Date	Mathematical invention
2000 B.C.	Computer algorithms for calculating roots of polynomials and eigenvalues
800 A.D.	Babylonian number system (based on 60)
1800 A.D.	Invention of the number 0
1800 A.D.	Invention of complex plane (real and imaginary axes)
1900 A.D.	Widespread use of negative numbers
1950 A.D.	Invention of vectors