2.1 Cosine addition formula to prove formulas for $\sin^2(x)$ and $\cos^2(x)$. (Section 2.6.1).

Use the addition formula for $\cos(x + x)$ in equation (2.8) and equation (2.2) to show

$$\sin^2(x) \;=\; \frac{1 - \cos(2\,x)}{2} \qquad\qquad \cos^2(x) \;=\; \frac{1 + \cos(2\,x)}{2}$$

2.2 Optional: Proof of cosine addition theorem by sine addition theorem.

Use the sine addition formula $\sin(x{+}y) = \sin(x)\cos(y) + \sin(y)\cos(x)$ to show

$$\sin(\underbrace{\alpha + \beta} \;+\; 90^\circ) \;=\; \cos(\alpha + \beta) \qquad\qquad \sin(\beta + 180^\circ) \;=\; \text{-}\sin(\beta)$$

Note: Setting $\alpha = 0$ in the first expression produces $\sin(\beta + 90^\circ) = \cos(\beta)$.
Combining the first and second expression, one can see $\sin(90^\circ + \beta \;+\; 90^\circ) \;=\; \cos(90^\circ + \beta) \;=\; \text{-}\sin(\beta)$.
Use $\sin(\alpha \;+\; \underbrace{\beta + 90^\circ}) \;=\; \cos(\alpha + \beta)$ and the previous facts to prove the cosine addition formula

$$\boxed{\cos(\alpha + \beta) \;=\; \cos(\alpha)\,\cos(\beta) \;-\; \sin(\alpha)\,\sin(\beta)}$$

2.3 Amplitude-phase examples. (See Section 2.6.3).

Use the ***amplitude/phase formulas for sine or cosine*** to complete the following blanks.

$$\text{-}\sin(2t) \;=\; \sin(2t + \boxed{}) \;=\; \cos(2t + \boxed{})$$

$$\text{-}\cos(2t) \;=\; \sin(2t + \boxed{}) \;=\; \cos(2t + \boxed{})$$

$$\text{-}3{*}\sin(2t) \;+\; 4{*}\cos(2t) \;=\; 5{*}\sin(2t + \boxed{})$$

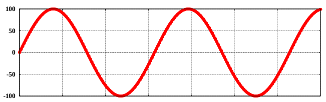
$$\text{-}3{*}\sin(2t + \frac{\pi}{3}) \;+\; 4{*}\cos(2t) \;=\; \boxed{}{*}\cos(2t + 0.82)$$

2.4 Design of noise cancelling headphones. (Section 2.6.3).

Noise cancelling headphones use both passive (insulated earphones) and active (electronic "anti-noise") methods to nullify ambient noise and create silence. One task of a sound engineer is to design low-energy anti-noise signals that help cancel ambient noise.

Courtesy Betai Koffi

Consider ambient noise of amplitude 100 and frequency ω (graphed right) of the form:



$$\text{AmbientNoise} \;=\; 100\,\sin(\omega\,t)$$

AmbientNoise is to be combined with $\text{AntiNoise} \;=\; A\,\sin(\omega\,t \;+\; \phi)$ whose amplitude A is a positive number.

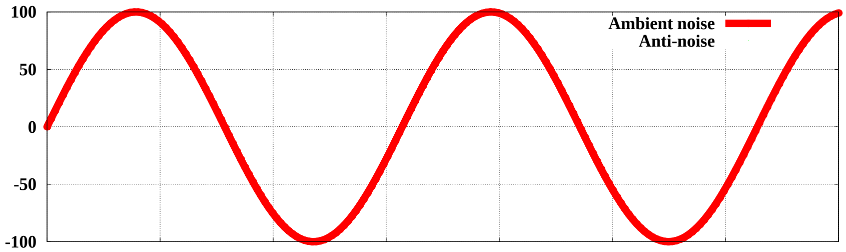
• Your sound-engineering task is to design AntiNoise so the sum AmbientNoise + AntiNoise has a combined amplitude of 10 (much quieter than AmbientNoise). Choose the phase ϕ that minimizes A (minimum A decreases hearing fatigue and energy consumption). On the plot, sketch your anti-noise signal.

Result:

$$\text{AmbientNoise} \;=\; 100\,\sin(\omega\,t)$$

$$\text{AntiNoise} \;=\; A\,\sin(\omega\,t + \phi)$$

$$A = \boxed{} \qquad \phi = \boxed{}\ \text{rad}$$



• It is difficult to design AntiNoise to be perfectly out of phase with AmbientNoise. Consider

AntiNoise $=$ 100 $\sin(\omega\, t \,+\, \pi \,+\, \delta)$. Determine the maximum δ between 0 and π to create a combined noise/anti-noise signal of amplitude 10, i.e.,

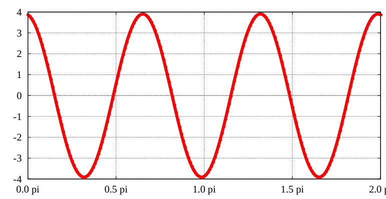
$$\text{Sound} \;=\; 100\,\sin(\omega\, t) \;+\; 100\,\sin(\omega\, t + \pi \,+\, \delta) \;=\; 10\,\sin(\omega\, t \,+\, \text{SomePhase})$$

Show δ is governed by the following equation – and solve for δ.

Result: $\sqrt{2 - 2\,\cos(\delta)} \;=\; 0.1$ $\qquad\qquad$ $\delta \,\approx\, 0.1$ rad

2.5 Manipulating sine and cosine functions into amplitude-phase form. (See Section 2.6.3).

An engineer is designing a wave-pool for a water park. The waves are created by two wave generators, and the amplitude, frequency, and phase of each wave at the center of the pool is measured separately and determined as shown below. With both wave generators operating, the resultant wave height $y(t)$ at the center of the pool, is approximately the sum of the two waves.



$$\text{wave}_A(t) \;=\; 4 * \sin(3 * t \;+\; \frac{\pi}{9}) \qquad \text{Note: } \frac{\pi}{9} \;=\; 20^\circ$$

$$\text{wave}_B(t) \;=\; 5 * \cos(3 * t \;+\; \frac{\pi}{3}) \qquad \text{Note: } \frac{\pi}{3} \;=\; 60^\circ$$

$$y(t) \;\approx\; \text{wave}_A(t) \;+\; \text{wave}_B(t)$$

First using intuition (guessing) and ***then*** using mathematics, determine $y(t)$ when it is expressed in terms of the (non-negative) amplitude C, frequency ω, and phase ϕ as $y(t) = C\,\sin(\omega * t \,+\, \phi)$. Comment on the accuracy of your intuition in predicting the wave's amplitude.

Result:

Intuition (**guess**):	$y(t) \;=\; \_\_\_ * \sin(\_\_\_ * t \;+\; \_\_\_)$
Mathematics (**analysis**):	$y(t) \;=\; \_\_\_ * \sin(\_\_\_ * t \;+\; \_\_\_)$
Mathematics (**analysis**):	$y(t) \;=\; \_\_\_ * \cos(\_\_\_ * t \;+\; 0.15)$
Accuracy in guessing wave amplitude:	**Good/Bad**

2.6 Is Newton's law $\vec{\mathbf{F}} = \frac{d\,(m\,\vec{\mathbf{v}}\,)}{dt}$ or $\vec{\mathbf{F}} = m\,\frac{d\vec{\mathbf{v}}}{dt}$? (Separate variables & integrate – Chapter 4).

The figure below shows a rocket moving with velocity $v\,\widehat{\mathbf{n}}_\mathrm{y}$. The rocket's propulsion system is malfunctioning and it ejects equal amounts of propellant left and right. The velocity of each propellant particle on the right is $2\widehat{\mathbf{n}}_\mathrm{x} + v\widehat{\mathbf{n}}_\mathrm{y}$ and each particle on the left is $-2\widehat{\mathbf{n}}_\mathrm{x} + v\widehat{\mathbf{n}}_\mathrm{y}$ (the $\widehat{\mathbf{n}}_\mathrm{y}$ component of the ejected propellant's velocity is **equal** to the rocket's velocity).

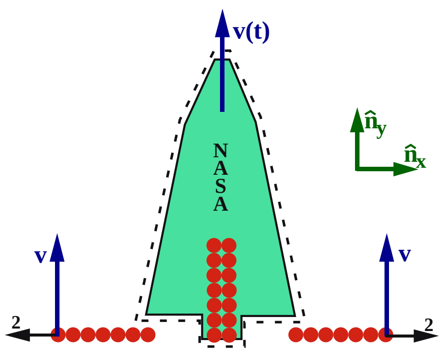
The two candidate motion laws shown below are suggested for analyzing the motion of the enclosed rocket system (the matter inside the dotted line in the figure below).[1]

$$\text{Is} \qquad \vec{\mathbf{F}} = \frac{d\,(m\,\vec{\mathbf{v}})}{dt} \qquad \textit{or} \qquad \vec{\mathbf{F}} = m\,\frac{d\,\vec{\mathbf{v}}}{dt}$$

Using each candidate equation, the speed $v(t)$ and kinetic energy K of the rocket at time t can be expressed in terms of the rocket's initial mass m_o and initial speed v_o. In the limit as $m \to 0$, these laws produce radically different values of v. Decide which provide more sensible results.[2] [3]

Note: There are no gravitational, electromagnetic, or other external forces on the system. Hence $\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

$$\begin{aligned} \vec{\mathbf{F}} &= \frac{d\,(m\,\vec{\mathbf{v}})}{dt} \\ \vec{\mathbf{0}} &= \frac{d\,(m\,\vec{\mathbf{v}})}{dt} \\ m\,\vec{\mathbf{v}} &= \overrightarrow{\textbf{constant}} = m_o\,v_o\,\widehat{\mathbf{n}}_\mathrm{y} \\ v(t) &= \frac{\boxed{}}{\boxed{}} \\ K &= \frac{1}{2}\,\frac{(m_o\,v_o)^2}{m} \end{aligned}$$



Rocket losing mass

$$\begin{aligned} \vec{\mathbf{F}} &= m\,\frac{d\vec{\mathbf{v}}}{dt} \\ \vec{\mathbf{0}} &= m\,\frac{d\vec{\mathbf{v}}}{dt} \\ \vec{\mathbf{0}} &= \frac{d\vec{\mathbf{v}}}{dt} \\ v(t) &= \boxed{} \\ K &= \frac{1}{2}\,m\,v_o^2 \end{aligned}$$

Law that results in v increasing as m decreases.	$\vec{\mathbf{F}} =$
Law that results in constant v.	$\vec{\mathbf{F}} =$
Law that conserves translational momentum of combined system (inside and outside dotted line).	$\vec{\mathbf{F}} =$
Based on this thought-experiment, the more plausible fundamental law.	$\vec{\mathbf{F}} =$



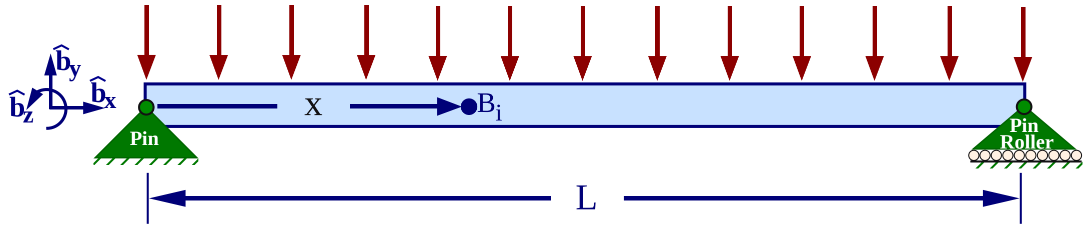
[1] Relativity is not relevant for this rocket. The fastest human object is the unmanned spacecraft Helios 2 (1976). Its peak speeds of $252,792\ \frac{\text{km}}{\text{hr}}$ (157,078 mph) are a mere $\frac{1}{4269}$ (0.0002) of the speed of light.

[2] A senior Lockheed-Martin dynamics engineer recently notified NASA of their incorrect version of Newton's second law on several NASA's websites, including http://exploration.grc.nasa.gov/education/rocket/newton2r.html and http://www.grc.nasa.gov/WWW/BGH/newton2.html. The correct law (according to Kane and others) is $\vec{\mathbf{F}} = m\,\frac{d\vec{\mathbf{v}}}{dt}$.

[3] There are other ways to properly account for the inflow and outflow of translational momentum.

2.7 Beam bending moment diagram via 2^{nd}-order ODE with boundary-values.

The following shows a long horizontal uniform elastic beam whose left end is connected to ground by an pin and whose right end is connected to ground by an pin-roller.



Note: x is the $\hat{\mathbf{b}}_x$ measure of B_i's position from the left-pin and is regarded as an independent variable.

Note: $M_z(x)$ is the internal "bending-moment" in the beam and is regarded as a dependent variable.

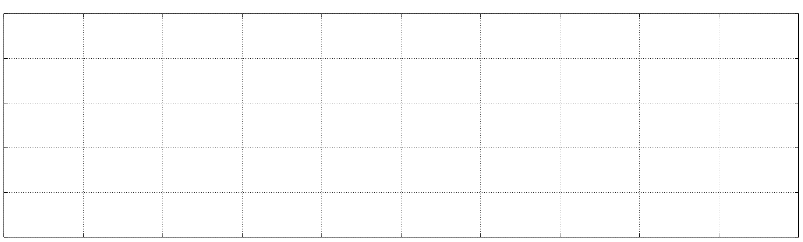
(a) One can use statics and ***free-body diagrams*** to relate shear force F_y via an integral (or the differential equation shown below) to a **downward**-vertical load per unit length $w(x)$ applied to the beam. Similarly, the bending moment M_z is related via an integral (or differential equation) to shear force F_y. Concatenate these relationships to relate M_z to $w(x)$.

Result: $\frac{d\,F_y}{dx} = w(x)$ $\qquad \frac{d\,M_z}{dx} = -F_y(x) \qquad \Rightarrow \qquad \frac{d^2\,M_z}{dx^2} =$ ______

(b) **Solve the ODE for M_z with separation of variables and integration.** **(See Section 4.1)** When the pin and pin-roller are ideal frictionless connections, the ***boundary values*** are $M_z(0) = 0$ and $M_z(L) = 0$ (L is the distance between the pin and pin-roller). Use these boundary values, a ***constant*** w, and the previous ODE to find an analytical expression for $M_z(x)$.

Result: (in terms of L, w, and x) $\qquad M_z(x) =$ ______

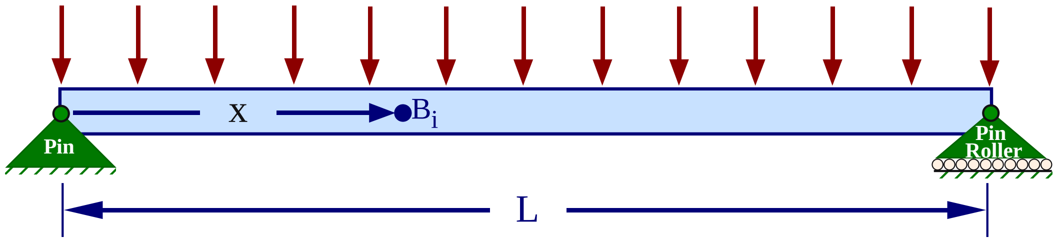
(c) **Sketch the bending-moment diagram** **[i.e., sketch $M_z(x)$ for $0 \le x \le L$].**



(d) The maximum value of M_z and where the beam is most likely to break is at $x =$ ______.

2.8 Beam deflection boundary-value problem - classification. (Section 3.5).

The following shows a long horizontal uniform elastic beam whose left end is connected to ground by a pin and whose right end is connected to ground by a pin-roller. The beam's shear, moment, slope, and deflection at an arbitrary point B_i on the beam's centerline depend on load, geometry, and material – and can be regarded as a function of two independent variables, namely time t and x.



Note: t (time) and x (horizontally-right measure of B_i's position along the beam) are regarded as independent variables.
Note: $v(x,t)$ is the vertically-upward deflection of the beam's elastic curve and is regarded as a dependent variable.

(a) The ***dynamic*** behavior of B_i's deflection is approximated by ***Euler beam theory*** with

$$\frac{\partial^2}{\partial x^2}\left(E\,I\,\frac{\partial^2 v}{\partial x^2}\right) \;+\; \rho\,\frac{\partial^2 v}{\partial t^2} \;=\; -w(x,t)$$

$w(x,t)$	Downward-vertical load per unit length
$E(x)$	Elastic modulus (material property)
$I(x)$	Centroidal area moment of inertia
$\rho(x)$	Mass density per unit length

Note: w, $E(x)$, $I(x)$, $\rho(x)$ are **known** (specified/prescribed) functions, whereas $v(x,t)$ is **unknown**.

Classify the previous equation by picking the relevant qualifiers from the following list.

Linear	**Homogeneous**	**Constant-coefficient**	**2^{nd}-order**	**Ordinary**	**Algebraic**
Nonlinear	**Inhomogeneous**	**Variable-coefficient**	**4^{th}-order**	**Partial**	**Differential**

(b) Simplify the previous equation to govern the beam's **static** deflection $v(x)$.
Next, classify the simplified equation by picking the relevant qualifiers from the list that follows.
Result: $\qquad = \qquad$

Linear	**Homogeneous**	**Constant-coefficient**	**2^{nd}-order**	**Ordinary**	**Algebraic**
Nonlinear	**Inhomogeneous**	**Variable-coefficient**	**4^{th}-order**	**Partial**	**Differential**

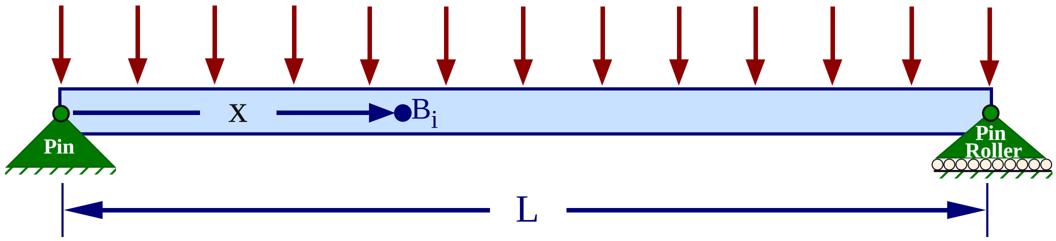
(c) Further simplify the previous equation to a uniform beam (**constant** E, I, and ρ).
Next, classify the simplified equation by picking the relevant qualifiers from the list that follows.
Result: $\qquad = \qquad$

Linear	**Homogeneous**	**Constant-coefficient**	**2^{nd}-order**	**Ordinary**	**Algebraic**
Nonlinear	**Inhomogeneous**	**Variable-coefficient**	**4^{th}-order**	**Partial**	**Differential**

(d) Using engineering intuition (guessing), I expect the maximum beam deflection and where the beam is most likely to break (under uniform load, i.e., w is **constant**) to be at $x =$ ________.

2.9 Beam deflection via 4^{th}-order ODE with boundary-values (separate variables & integrate).

The following shows a long horizontal uniform elastic beam of length L whose left end is connected to ground by an ideal frictionless pin and whose right end is connected to ground by an ideal frictionless pin-roller. The beam's shear, moment, slope, and deflection at an arbitrary point B_i on the beam's centerline depend on load, geometry, and material and are a function of x.



Note: x is a horizontally-right measure of B_i's position along the beam and is regarded as an independent variable.
Note: $v(x)$ is the vertically-upward deflection of the beam's elastic curve and is regarded as a dependent variable.

This problem calculates **static** deflection of $v(x)$ (vertically-downward measure of B_i's deflection) of a beam whose elastic modulus E and centroidal area moment of inertia I are **known constants**. A **known constant** downward-vertical force per unit length w is applied to the beam.

(a) Using ***separation of variables and integration***, solve the following 4^{th}-order ODE for $v(x)$.

$$\textit{Euler beam equation:} \qquad E\,I\,\frac{d^4v}{dx^4} = -w$$

Result: (In terms of x, the *constants* E, I, w, and arbitrary constants c_0, c_1, c_2, c_3)

$$v(x) = \boxed{}$$

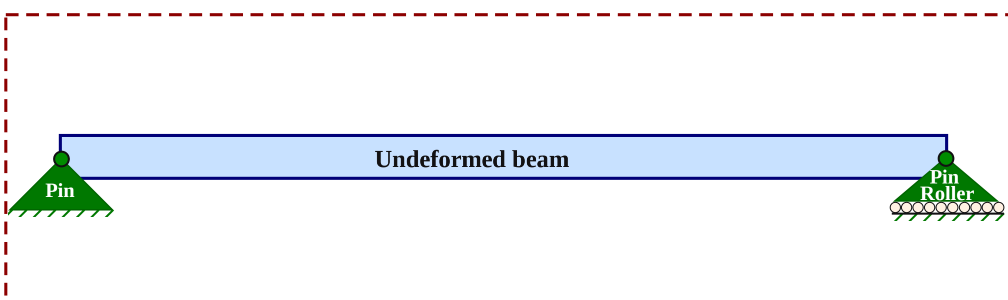
(b) The values of the constants c_0, c_1, c_2, c_3 depend on ***boundary values***. The displacement boundary values are $v(0) = 0$ and $v(L) = 0$. When the pin and pin-roller are ideal frictionless connections, they cannot support moments, hence two ***boundary values*** are $M(0) = 0$ and $M(L) = 0$. Using these moment boundary values, the relationship $M = E\,I\,\frac{d^2v}{dx^2}$, find an analytical expression for $v(x)$ in terms of L, E, I, w, and x.

Result:

$$v(x) = \frac{-w}{24\,E\,I} * \left(\boxed{} - 2\,\boxed{}\,x^3 + \boxed{}\,x \right)$$

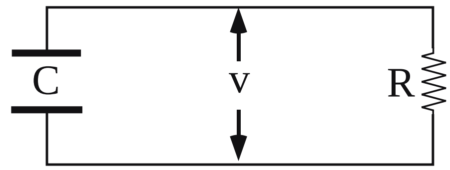
(c) Sketch the shape of the **deformed** beam for $0 \le x \le L$.

Result:



2.10 Solve an electrical circuit's 1^{st}-order ODE (separate variables & integrate).

The following "R-C" circuit has constant linear resistance R and constant linear capacitance C. Equating the electrical current i through the resistor and capacitor yields an ODE for voltage v across the resistor/capacitor.



$$i \;=\; \frac{v}{R} \;=\; C\,\frac{dv}{dt} \qquad \Rightarrow \qquad C\,\frac{dv}{dt} \;+\; \frac{1}{R}\,v \;=\; 0$$

(a) Classify the ODE for v by picking the relevant qualifiers from the following list.

Linear	**Homogeneous**	**Constant-coefficient**	**1^{st}-order**	**Algebraic**
Nonlinear	**Inhomogeneous**	**Variable-coefficient**	**2^{nd}-order**	**Differential**

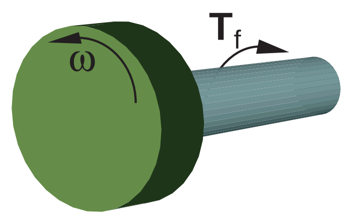
(b) Using ***separation of variables and integration***, find the solution for $v(t)$.
Result: [in terms of C, R, t, and v_0, the ***initial value*** of v, i.e., $v_0 \triangleq v(t{=}0)$].

$$v(t) \;=\;$$

(c) Design R and C so $i = 2.0$ milliAmps and $v = 40$ volts at $t = 2.0$ sec. Use $v_0 = 120$ volts.
Result: $R \;=\; 20$ kOhms (20,000 Ohms) $\qquad C \;=\; 91$ microFarads (9.1×10^{-5} Farads)

2.11 System identification for a 1^{st}-order dynamic system (Coulomb friction).

The figure to the right shows a free-spinning (non-driven) motor whose laboratory data for spin rate ω correlates with the mathematical function

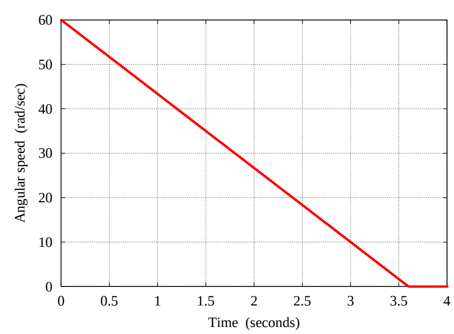


$$\omega(t) \;=\; \omega_0 \;+\; slope * t$$

where $slope$ is the slope of the line. and ω_0 is the ***initial spin rate*** (i.e, ω_0 is the value of ω at $t = 0$). This mathematical function suggests a 1^{st}-order ODE of the form[a]

$$I\,\frac{d\omega}{dt} \;=\; -T_f$$

where I is a constant (the moment of inertia of the motor and its rotor and attachments about the motor's axis) and T_f is the **constant** torque exerted by Coulomb friction.



[a] With knowledge of ODEs, an engineer can view sensor data and decide if the motor is dominated by Coulomb friction, viscous friction, etc.

(a) Classify the previous equation by picking the relevant qualifiers from the following list.

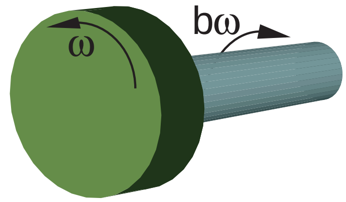
Linear	**Homogeneous**	**Constant-coefficient**	**1^{st}-order**	**Algebraic**
Nonlinear	**Inhomogeneous**	**Variable-coefficient**	**2^{nd}-order**	**Differential**

(b) Using ***separation of variables and integration***, find $\omega(t)$ in terms of I, T_f, t, and ω_0.
Result:

$$\omega(t) \;=\;$$

(c) A motor having an inertia (with attached rotor) of $I = 300\ \text{g cm}^2$ is spinning with $\omega = 60$ rad/sec. The power is then shut off and the motor spins down with the time-history shown in the previous graph. Approximate T_f, the Coulomb friction torque constant.
Result: $T_f \;\approx\; 0.5$ milliNewtons $*$ m

2.12 System identification for a 1^{st}-order dynamic system (viscous damping).



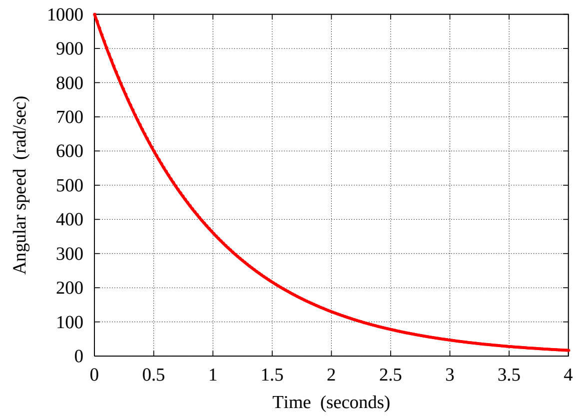
The figure to the right shows a freely-spinning (non-driven) motor whose laboratory data correlates with the mathematical function

$$\omega(t) \;=\; \omega_0 \, e^{\frac{-t}{\tau_c}}$$

where ω_0 is the initial angular spin rate and τ_c is called the ***time constant***. This mathematical function suggests a 1^{st}-order ODE of the form

$$I \frac{d\omega}{dt} \;+\; b\,\omega \;=\; 0$$

where I is a constant (the moment of inertia of the motor and its rotor and attachments about the motor's axis) and b is the linear viscous damping constant.



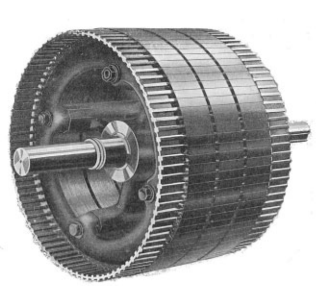
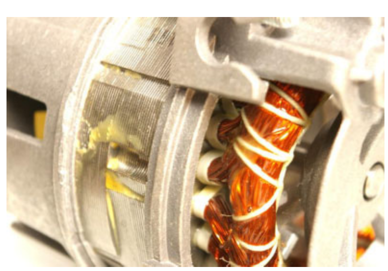
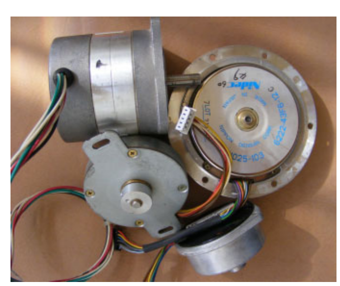
(a) Classify the previous equation by picking the relevant qualifiers from the following list.

Linear	**Homogeneous**	**Constant-coefficient**	**1^{st}-order**	**Algebraic**
Nonlinear	**Inhomogeneous**	**Variable-coefficient**	**2^{nd}-order**	**Differential**

(b) A motor is spinning with $\omega = 1000$ rad/sec. The power is shut off and the motor spins down with the time-history shown. Use the laboratory data to approximate the ***time-constant*** τ_c.
Result: $\tau_c \;\approx\; 0.98$ sec Note: ***time constant*** τ_c is defined in Section 4.2.

(c) A motor having inertia (with attached rotor) of $I = 300 \text{ g cm}^2$ is spinning with $\omega = 1000$ rad/sec. The power is then shut off and the motor spins down. By observing the time-history of the angular speed of the motor, an engineer approximates the time-constant to be $\tau_c = 4$ sec. Assuming that linear viscous damping is the only relevant factor in the motor's spin-down, determine the linear viscous damping constant b.[4]
Result:

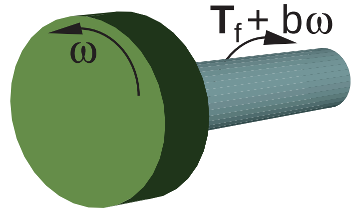
$$b \;=\; 75 \, \frac{\text{g} * \text{cm}^2}{\text{sec}}$$

[4]Hint: Use ***separation of variables and integration***, to solve for $\omega(t)$ in terms of I, b, and ω_0 (the ***initial value*** of ω).

2.13 System identification for a 1^{st}-order dynamic system (Coulomb friction and viscous damping).

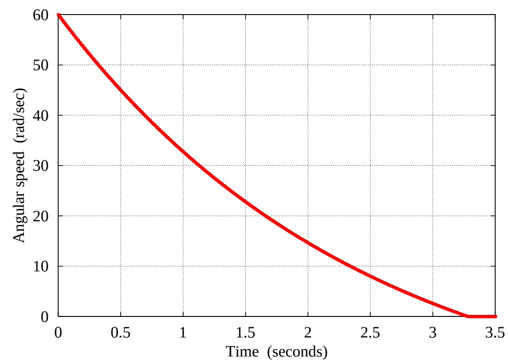
The figure to the right shows a freely-spinning (non-driven) motor. Its laboratory data suggests that both Coulomb friction and linear viscous damping act on the motor. The equation governing ω (the motor's angular speed) is



$$I\,\dot{\omega} \;+\; b\,\omega \;=\; \text{-}T_f$$

- I is the moment of inertia of the motor and its attachments about the motor's axis
- b is the linear viscous damping constant
- T_f is the constant torque exerted by Coulomb friction

The graph indicates that both Coulomb friction and viscous damping are present. At large values of ω, viscous damping dominates and motor speed decreases exponentially whereas at small values of ω, Coulomb friction dominates and motor speed decreases linearly.



(a) Classify the previous equation by picking the relevant qualifiers from the following list.

Linear	**Homogeneous**	**Constant-coefficient**	**1^{st}-order**	**Algebraic**
Nonlinear	**Inhomogeneous**	**Variable-coefficient**	**2^{nd}-order**	**Differential**

(b) Use ***separation of variables*** to show the ODE can be written as shown (below left). Next, ***integrate*** to solve for $\omega(t)$ in terms of I, b, T_f, and ω_0 (the initial value of ω).
Result: Note: It may be helpful to make a substitution ($u = \ldots$) to do the integral.

$$\frac{I\,d\omega}{b\,\omega \,+\, T_f} \;=\; \text{-}dt \qquad\qquad \Rightarrow \qquad\qquad \omega(t) \;=\; \boxed{}$$

Note: Substituting $T_f = 0$ gives $\omega(t) \;=\; \omega_0\, e^{\frac{-b}{I}\,t}$ which is useful for Homework 2.12. Doing a Taylor-series expansion about $b\!=\!0$ in the solution for $\omega(t)$ gives $\omega(t) \;=\; \omega_0 \;-\; \frac{T_f}{I}$ which verifies the solution in Homework 2.11.

(c) A motor having inertia (with attached rotor) of $I = 1.0\ \text{kg m}^2$ is spinning with initial angular speed $\omega_0 = 60$ rad/sec. The power is then shut off and the motor spins down with the time-history shown above. Using the laboratory data, form two equations that suffice to approximate b and T_f. Two convenient points for accurate results are $\omega(t = 0.5) \approx 45$ and $\omega(t = 2.5) \approx 8$.[5]
Next, classify those two equations by picking the relevant qualifiers from the list that follows.
Result:

$$(60 + \frac{T_f}{b}) * e^{-0.5\,b} \;-\; \frac{T_f}{b} \;-\; 45 \;=\; 0$$

$$(60 + \frac{T_f}{b}) * e^{-2.5\,b} \;-\; \frac{T_f}{b} \;-\; 8 \;=\; 0$$

Uncoupled	**Linear**	**Homogeneous**	**Algebraic**
Coupled	**Nonlinear**	**Inhomogeneous**	**Differential**

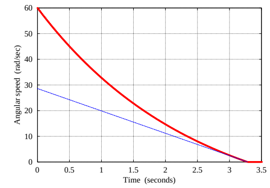
With a pencil and paper, can *you* solve these equations for b and T_f? **Yes/No**.

(d) Calculate b and T_f (e.g., use MATLAB®, WolframAlpha, or **M**otion**G**enesis with the function Exp() for e and submit a print-out of your file or a screen shot).
Hint: **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Solve nonlinear algebraic equations.**

[5] Two data points are needed to form two equations to approximate b and T_f. Since resisting torque is modeled $T_f + b\,\omega$, use a first data point when the motor spins quickly (ω is large so linear viscous damping $b\,\omega$ dominates), and a second data point when the motor spins slowly (ω is small so Coulomb friction T_f dominates).

Result: $b = 0.41 \dfrac{\text{kg}\,\text{m}^2}{\text{sec}}$ $\qquad T_f = 8.72\ \text{N}\,\text{m}$

Note: An estimate of T_f is the slope of the tangent line as $\omega \to 0$.



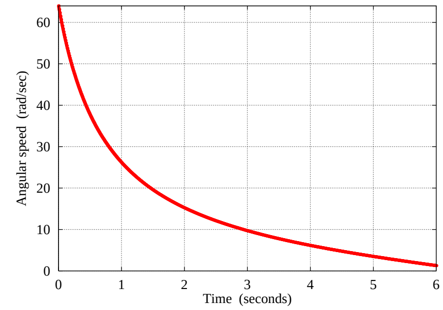
2.14 Optional: Determining damping and friction constants from experimental data.

The following ODE (below-right) governs the angular speed ω of a motor whose motor and attached rotor have a moment of inertia of $I = 1.0\ \text{kg}\,\text{m}^2$. The associated graph shows the time-history of ω for a spin-down test corresponding to an initial value of $\omega(t=0) = 64$ rad/sec.

Laboratory data for $\omega(t)$

Time (sec)	Angular speed ω (rad/sec)
0.5	37.74139
2.0	15.23263
4.0	6.147114

$$I \frac{d\omega}{dt} = -\left(T_f + b\,\omega + c\,\omega^2\right)$$



Solve the ODE for $\omega(t)$ with separation of variables.
Next, use the laboratory data to approximate T_f, b, c.

Result: $\omega(t) =$

$T_f = 2.0\ \text{N}\,\text{m} \qquad b = 0.04\ \text{N}\,\text{m}\,\text{s} \qquad c = 0.02\ \text{N}\,\text{m}\,\text{s}^2$

2.15 System dynamics for a parachutist.

The figure to the right shows a parachutist of mass $m = 100$ kg falling at speed v near Earth whose gravitational acceleration is $g = 9.8\ \frac{\text{m}}{\text{s}^2}$. The parachute provides upward air-resistance force of magnitude $b\,v$ (b is a constant). An equation for v is formed as:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} \qquad \Rightarrow \qquad m\,\dot{v} + b\,v = m\,g$$

Courtesy Jorge Cham

Pertinent questions for a parachutist are:
"How fast will I hit the ground?".
"How much time will it take to hit ground?".

(a) Classify the previous equation by picking the relevant qualifiers from the following list.

Linear	**Homogeneous**	**Constant-coefficient**	**1^{st}-order**	**Algebraic**
Nonlinear	**Inhomogeneous**	**Variable-coefficient**	**2^{nd}-order**	**Differential**

(b) ***Separation of variables and integration*** (and use of initial value).
Separate: Determine functions $f(v)$ and $g(t)$ so the ODE can be written as $f(v)\,dv = g(t)\,dt$.
Integrate: Solve the previous equation for $v(t)$ in terms of m, b, g, and t when $v(t{=}0) = 0$.
Result: $\boxed{}\ dv = \boxed{}\ dt \qquad \Rightarrow \qquad v(t) = \boxed{}$

(c) Referring to Section 10.5, determine the ***steady-state*** part and ***transient part*** of $v(t)$ or explain why you are unable to find them or what additional information is needed.
Result:
$v_{\text{ss}}(t) \underset{(10.7)}{\triangleq} \lim\limits_{t\to\infty} v(t) = \boxed{} \qquad v_{\text{transient}}(t) \underset{(10.7)}{\triangleq} v(t) - v_{\text{ss}}(t) = \boxed{}$

(d) ***Separation of variables and integration*** (and use of initial value).
The vertical downward distance that the parachutist falls is denoted $y(t)$.
Find $y(t)$ in terms of m, b, g, and t when $y(t{=}0) = 0$.
Result:

$$y(t) \;=\; \text{____} \;+\; \text{____} \;+\; \frac{m^2\, g}{b^2}\, e^{\frac{-b}{m}\, t}$$

(e) Referring to Section 10.5, determine the ***steady-state*** part and ***transient part*** of $y(t)$ or explain why you are unable to find them or what additional information is needed.
Result:

$$y_{\text{ss}}(t) \;\underset{(10.7)}{\triangleq}\; \lim_{t \to \infty} y(t) \;=\; \text{____}$$

$$y_{\text{transient}}(t) \;\underset{(10.7)}{\triangleq}\; y(t) \;-\; y_{\text{ss}}(t) \;=\; \text{____}$$

(f) Knowing the parachutist's ***terminal velocity*** is 4.9 $\frac{\text{m}}{\text{s}}$ ($v = 4.9\ \frac{\text{m}}{\text{s}}$ after the parachute has been open enough time), form an exact equation for t_f (the time for a 100 kg parachutist to fall a distance y_f).
Result: $b \;=\; 200\ \frac{\text{N s}}{\text{m}}$ $\qquad y_f \;=\; -2.45 \;+\; 4.9\, t_f \;+\; \text{____}\, e^{-2\, t_f}$

(g) Classify the previous equation by picking the relevant qualifiers from the following list.

Linear	**Homogeneous**	**Algebraic**
Nonlinear	**Inhomogeneous**	**Differential**

In ***general***, equations classified like this are solved with ____.

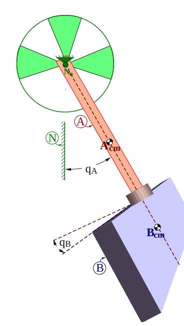
(h) Use engineering/mathematical insights to approximate t_f in terms of y_f.
What conditions make your approximation better?
Result:
$t_f \;\approx\;$ ____ ____

(i) ***Separation of variables for 2nd-order ODEs?***
Separation of variables is useful for solving 1st-order ODEs. Is it possible to find an analytical solution for $y(t)$ for the 2nd-order ODE to the right via separation of variables. **Yes/No**.

$$\frac{d^2y}{dt^2} \;+\; 2\,\frac{dy}{dt} \;=\; 9.8$$

Explain: ____

2.16 Numerical solution of ordinary differential equations.



The equations of motion for the system described in Section 3.1 are

$$\ddot{q}_A \;=\; \frac{2\;[508.89\,\sin(q_A) \;-\; \sin(q_B)\,\cos(q_B)\,\dot{q}_A\,\dot{q}_B]}{-21.556 + \sin^2(q_B)}$$

$$\ddot{q}_B \;=\; -\sin(q_B)\,\cos(q_B)\,\dot{q}_A^2$$

(a) Classify the previous set of equations by picking the relevant qualifiers from the list below.

Uncoupled	**Linear**	**Homogeneous**	**Constant-coefficient**	**1st-order**	**Algebraic**
Coupled	**Nonlinear**	**Inhomogeneous**	**Variable-coefficient**	**2nd-order**	**Differential**

(b) **Optional:** Find the maximum absolute value of $q_B(t)$ for $0 \le t \le 10$. Use $q_B(0) = 1.0°$ and two different initial values of q_A, namely $q_A(0) = 45°$ and $q_A(0) = 90°$. Submit a printed copy of your **M**otion**G**enesis or MATLAB® file. Hint: www.MotionGenesis.com ⇒ Get Started ⇒ **Solve ODEs.**
Result:
- Maximum absolute value of $q_B(t)$ when $q_A(0) = 45°$: $\quad q_B(t{=}1.66) \;=\;$ ____$°$
- Maximum absolute value of $q_B(t)$ when $q_A(0) = 90°$: $\quad q_B(t{=}8.18) \;=\;$ ____$°$