3.4 Independent, dependent, and specified variables

To classify a differential equation, four types of mathematical quantities need to be identified.

- An ***independent variable*** is a quantity that varies independently, i.e., it is not controlled by (or depend on) other variables.[1] For many dynamic systems, there is one independent variable that is either t (time) or s (a complex variable used in conjunction with the Laplace transform).

- The symbol x is a ***dependent variable*** if its value depends on the independent variable and its dependence is considered to be **unknown**, e.g., x is governed by a differential equation.

- The symbol x is a ***specified variable*** if its dependence on the independent variable can be considered to be **known**. For example, x is specified (or ***prescribed***) when $x = sin(t)$ is **known** apriori.

- The symbol c is a ***constant*** if its value does not change, i.e., its value does not depend on the independent variable.

3.5 Classification of algebraic and differential equations

Uncoupled	Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

In the following examples, x and y are dependent variables and t is the independent variable time.[2]

1. **Differential and algebraic equations**
   In each example below, circle ***differential*** if the equation contains a time-derivative of a dependent variable (unknown), otherwise circle ***algebraic***.

   If needed: Answers to these interactive questions are at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

   $$\frac{dy}{dt} = 7 \qquad\qquad \boxed{\textbf{differential}}/\textbf{algebraic} \text{ equation}$$
   $$3 * y = 7 \qquad\qquad \textbf{differential}/\boxed{\textbf{algebraic}} \text{ equation}$$
   $$\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial s} = 0 \qquad\qquad \boxed{\textbf{differential}}/\textbf{algebraic} \text{ equation}$$
   $$7y^4 + \cos(t)\sin(y)^3 = \tan(t) + 99 \qquad\qquad \textbf{differential}/\boxed{\textbf{algebraic}} \text{ equation}$$

   - **Ordinary and partial differential equations**
     When the dependent variable x is a function of only *one* independent variable t, the derivative of x with respect to t is called an ***ordinary derivative***,[3] and is denoted $\frac{dx}{dt}$ or $\dot{x}$ or x'. Alternatively, if x is a function of *two or more* independent variables (e.g., s and t), the derivative of x with respect to t is called the ***partial derivative*** of x with respect to t, and is denoted $\frac{\partial x}{\partial t}$.[4]
     In each example below, circle ***ordinary*** if the equation is an ***ODE***, i.e., contains only ordinary derivatives of dependent variables. Circle ***partial*** is the equation is an ***PDE***, i.e., contains a partial derivative of a dependent variable.

     $$\frac{dx}{dt} = e^t \qquad\qquad \boxed{\textbf{ordinary}}/\textbf{partial} \text{ differential equation}$$
     $$\frac{\partial y}{\partial t} = 0 \qquad\qquad \textbf{ordinary}/\boxed{\textbf{partial}} \text{ differential equation}$$
     $$\frac{d^2 x}{dt^2} + \sin(t)\left(\frac{dx}{dt}\right)^2 + \tan(x) = e^t \qquad\qquad \boxed{\textbf{ordinary}}/\textbf{partial} \text{ differential equation}$$
     $$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial s^2} = 0 \qquad\qquad \textbf{ordinary}/\boxed{\textbf{partial}} \text{ differential equation}$$

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[1]To help remember t is an ***independent*** variable, associate t with an ***independent***, uncontrollable, []teenager.

[2]It is helpful to make an association of **GOOD** or **BAD** with each qualifier, e.g., linear is **GOOD** and nonlinear is **BAD**.

[3]Mathematicians regard ***ordinary*** derivatives as ***plain and boring*** whereas equations with 2^+ independent variables and ***partial*** derivatives are "***hot and spicy***" .

[4]This textbook studies ***ODEs*** (ordinary differential equations), i.e., with **one** independent variable – not ***PDEs***.

- **First-order, second-order, third-order, ..., differential equations**
  The order of a differential equation is the order of the highest derivative of a dependent variable in the equation.[5]   Complete each blank below with 1st-**order**, 2nd-**order**, or 3rd-**order**.

  $$\sin(t)\left(\frac{dx}{dt}\right)^3 \;+\; \tan(x) \;=\; 0 \qquad\qquad \boxed{\mathbf{1^{st}}}\text{-order ODE}$$

  $$\frac{d^2x}{dt^2} \;+\; \left(\frac{dx}{dt}\right)^3 \;+\; \tan(x) \;=\; 0 \qquad\qquad \boxed{\mathbf{2^{nd}}}\text{-order ODE}$$

  $$\frac{\partial^3 y}{\partial t^3} \;+\; \frac{\partial y}{\partial s} \;=\; 0 \qquad\qquad \boxed{\mathbf{3^{rd}}}\text{-order PDE}$$

- **Constant, periodic, and variable coefficient differential equations**
  A differential equation is classified by the coefficients of factors containing dependent variables. Coefficients are either constant or depend exclusively on ***independent variables***, e.g.,[6]

Constant-coefficient	***All*** the coefficients are constants.
Periodic-coefficient	***All*** the coefficients are constants or periodic.
Variable-coefficient	***One*** or more coefficients are neither constant nor periodic.

  Complete each blank below with **constant**, **periodic**, or **variable**.

  $$\ddot{x} \;+\; 3*x \;-\; \sin(t) \;=\; 0 \qquad\qquad \boxed{\mathbf{constant}}\text{-coefficient, }2^{nd}\text{-order, ODE}$$

  $$\ddot{x} \;+\; t*x \;=\; 0 \qquad\qquad \boxed{\mathbf{variable}}\text{-coefficient, }2^{nd}\text{-order, ODE}$$

  $$3\ddot{x} \;+\; x*\dot{x} \;+\; 7*\sin(x)*x \;=\; 0 \qquad\qquad \boxed{\mathbf{constant}}\text{-coefficient, }2^{nd}\text{-order, ODE}$$

  $$\sin(t)*\dot{x} \;+\; x \;=\; 0 \qquad\qquad \boxed{\mathbf{periodic}}\text{-coefficient, }1^{st}\text{-order, ODE}$$

2. **Homogeneous and inhomogeneous (differential or algebraic equations)**
   A differential equation is classified by whether it not all its terms have a dependent variable.

Homogeneous	***All*** terms contain one or more dependent variables.
Inhomogeneous	***One*** term is constant or depends exclusively on ***independent variables***

   Complete each example below by circling **homogeneous** or **inhomogeneous**.

   $$x \;+\; y \;=\; 0 \qquad\qquad \boxed{\textbf{homogeneous}}/\textbf{inhomogeneous} \text{ algebraic equation}$$

   $$x \;+\; y \;=\; t \qquad\qquad \textbf{homogeneous}/\boxed{\textbf{inhomogeneous}} \text{ algebraic equation}$$

   $$t^2\,\ddot{x} \;+\; \sin(t)\,\dot{x}^2 \;+\; x \;=\; 0 \qquad\qquad \boxed{\textbf{homogeneous}}/\textbf{inhomogeneous} \text{ ODE}$$

   $$\ddot{x} \;+\; 3 \;=\; 0 \qquad\qquad \textbf{homogeneous}/\boxed{\textbf{inhomogeneous}} \text{ ODE}$$

   Note: Equations are frequently written in a "standard form" with all the **unknowns** (e.g., terms containing x, $\dot{x}$, $\ddot{x}$, y, $\dot{y}$, $\ddot{y}$) on the left-hand side of the equal sign and all the remaining terms on the right-hand side. The equation's right-hand side will either be 0, a constant, or exclusively a function of independent variables. If the right-hand side is 0, the equation is ***homogeneous*** otherwise it is ***inhomogeneous***.[7]

3. **Linear and nonlinear (differential or algebraic equations)**
   To determine if an expression is ***linear*** or ***nonlinear***, one must first ask *"Linear in what?"*. In the context of differential equations, a differential equation is ***linear*** if it is linear in its ***dependent variables*** and their derivatives (e.g., linear in x, $\dot{x}$, $\ddot{x}$, y, $\dot{y}$, $\ddot{y}$).

   For example, a ***linear*** 2nd-order ODE with one dependent variable x can be expressed as

   $$c_0 \;+\; c_1*x \;+\; c_2*\dot{x} \;+\; c_3*\ddot{x} \;=\; 0$$

   where c_i (i=0,1,2,3) are **not** functions of x, $\dot{x}$, $\ddot{x}$, (but may be functions of t).

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[5] A physical system that is governed by a nth-order differential equation is called an nth-order system.

[6] Periodic-coefficient ODEs are a special subset of variable-coefficient ODEs whose solution involves Floquet theory.

[7] Separating terms with a dependent variable from those without is analogous to a farmer separating sheep from goats. If the farmer only has sheep, the herd is ***homogeneous***. If, after the farmer has put all dependent variables into the barn, there are any terms still out in the fields, the equation is ***inhomogeneous***.

A **linear** 1st-order ODE with dependent variables x, y, z can be expressed as

$$c_0 \ + \ c_1 * x \ + \ c_2 * \dot{x} \ + \ c_3 * y \ + \ c_4 * \dot{y} \ + \ c_5 * z \ + \ c_6 * \dot{z} \ = \ 0$$

where c_i (i = 0, ..., 6) are **not** functions of x, $\dot{x}$, y, $\dot{y}$, z, $\dot{z}$, (but may be functions of t).

Complete each example below by circling **linear** or **nonlinear**.

$[t^2 + \sin(t)]\,x \ + \ \tan(t) \ = \ 4$	**linear**/nonlinear	inhomogeneous algebraic equation
$x^2 \ + \ \sin(x) \ = \ 4$	linear/**nonlinear**	inhomogeneous algebraic equation
$t^2\,\ddot{x} \ + \ \sin(t)\,\dot{x} \ + \ x \ = \ 0$	**linear**/nonlinear	homogeneous ODE
$\ddot{x} \ + \ \sin(x) \ = \ 0$	linear/**nonlinear**	homogeneous ODE

Note: Solving *nonlinear equations* is usually significantly *more difficult* than solving *linear equations*. Even determining the *number* of solutions to nonlinear equations can be difficult, e.g., a nonlinear equation may have no solution, 1 solution, 2 solutions, 17 solutions, or infinite solutions. For example, the solutions to $x^2 + \sin(x) = 4$ are $x \approx 1.736$ and $x \approx 2.194$, and can be obtained by a variety of methods, e.g., Newton-Rhapson or graphing.

4. **Uncoupled and coupled** (**differential or algebraic equations**)
   To determine if a set of equations is *coupled*, one must first ask *"Coupled in what?"*. A *set* of equations is *coupled* when the same unknown appears in more than one equation. A set of ODEs is coupled if a *dependent variable* or its derivative (e.g., x, $\dot{x}$, $\ddot{x}$) appear in more than one equation.

   Complete each blank below with *coupled* or *uncoupled*.

$t^2\,\cos(x^3) \ + \ \sin(t)\,x^2 \ + \ x \ = \ 0$ $y \ + \ t^3\,\sin(y) \ - \ 9 \ = \ 0$	**uncoupled**	nonlinear inhomogeneous algebraic
$x \ + \ y \ = \ 0$ $x \ = \ 7$	**coupled**	linear inhomogeneous algebraic
$t^2\ddot{x} \ + \ \sin(t)\dot{x}^2 \ + \ x \ = \ 0$ $\ddot{y} \ + \ t^3\,\sin(y) \ - \ \tan(t) \ = \ 0$	**uncoupled**	nonlinear inhomogeneous ODE
$\dot{x} \ + \ \dot{y} \ = \ 0$ $\ddot{x} \ = \ 0$	**coupled**	linear homogeneous ODE

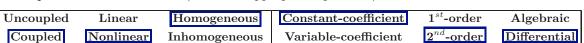## 3.6   Example of classification of differential equations

The equations governing the motion of the system described in Section 3.1 are

$$\ddot{q}_A \ = \ \frac{2\,[508.89\,\sin(q_A) \ - \ \sin(q_B)\,\cos(q_B)\,\dot{q}_A\,\dot{q}_B]}{-21.556 + \sin^2(q_B)}$$

$$\ddot{q}_B \ = \ -\sin(q_B)\,\cos(q_B)\,\dot{q}_A^2$$

This set of equations is classified as (circle the appropriate qualifiers)

Uncoupled	Linear	**Homogeneous**	**Constant-coefficient**	1st-order	Algebraic
Coupled	**Nonlinear**	Inhomogeneous	Variable-coefficient	**2nd-order**	**Differential**

It is worth noting that in general, an algebraic equation is **good** if it is linear. A differential equation is **good** if it is linear, homogeneous, constant-coefficient, first-order, and ordinary.[8]

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[8]A **good** equation means that it is easy to manipulate and obtain a solution. A **bad** equation means that instead of enjoying time with your friends, you are struggling with difficult mathematics.