3.4 Independent, dependent, and specified variables

To classify a differential equation, four types of mathematical quantities need to be identified.

- An **independent variable** is a quantity that varies independently, i.e., it is not controlled by (or depend on) other variables.\(^1\) For many dynamic systems, there is one independent variable that is either \(t\) (time) or \(s\) (a complex variable used in conjunction with the Laplace transform).

- The symbol \(x\) is a **dependent variable** if its value depends on the independent variable and its dependence is considered to be **unknown**, e.g., \(x\) is governed by a differential equation.

- The symbol \(x\) is a **specified variable** if its dependence on the independent variable can be considered to be **known**. For example, \(x\) is specified (or **prescribed**) when \(x = \sin(t)\) is **known** apriori.

- The symbol \(c\) is a **constant** if its value does not change, i.e., its value does not depend on the independent variable.

3.5 Classification of algebraic and differential equations

<table>
<thead>
<tr>
<th>Uncoupled</th>
<th>Linear</th>
<th>Homogeneous</th>
<th>Constant-coefficient</th>
<th>1(^{st})-order Algebraic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupled</td>
<td>Nonlinear</td>
<td>Inhomogeneous</td>
<td>Variable-coefficient</td>
<td>2(^{nd})-order Differential</td>
</tr>
</tbody>
</table>

In the following examples, \(x\) and \(y\) are dependent variables and \(t\) is the independent variable time.\(^2\)

1. **Differential and algebraic equations**

In each example below, circle **differential** if the equation contains a time-derivative of a dependent variable (unknown), otherwise circle **algebraic**.

Note: Answers to these interactive questions are at [www.MotionGenesis.com](http://www.MotionGenesis.com) ⇒ Textbooks ⇒ Resources.

\[
\frac{dy}{dt} = 7 \quad \text{differential/algebraic equation}
\]

\[
3y = 7 \quad \text{ordinary/algebraic equation}
\]

\[
\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial s} = 0 \quad \text{differential/algebraic equation}
\]

\[
7y^4 + \cos(t)\sin(y)^3 = \tan(t) + 99 \quad \text{differential/algebraic equation}
\]

- **Ordinary and partial differential equations**

When the dependent variable \(x\) is a function of only **one** independent variable \(t\), the derivative of \(x\) with respect to \(t\) is called an **ordinary derivative**,\(^3\) and is denoted \(\frac{dx}{dt}\) or \(\dot{x}\) or \(x'\). Alternately, if \(x\) is a function of **two or more** independent variables (e.g., \(s\) and \(t\)), the derivative of \(x\) with respect to \(t\) is called the **partial derivative** of \(x\) with respect to \(t\), and is denoted \(\frac{\partial x}{\partial t}\).\(^4\)

In each example below, circle **ordinary** if the equation is an **ODE**, i.e., contains only ordinary derivatives of dependent variables. Circle **partial** is the equation is an **PDE**, i.e., contains a partial derivative of a dependent variable.

\[
\frac{dx}{dt} = e^t \quad \text{ordinary/partial differential equation}
\]

\[
\frac{\partial y}{\partial t} = 0 \quad \text{ordinary/partial differential equation}
\]

\[
\frac{d^2 x}{dt^2} + \sin(t)\left(\frac{dx}{dt}\right)^2 + \tan(x) = e^t \quad \text{ordinary/partial differential equation}
\]

\[
\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial s^2} = 0 \quad \text{ordinary/partial differential equation}
\]

\(^1\)To help remember \(t\) is an **independent** variable, associate \(t\) with an **independent**, uncontrollable, \(\text{[teenager.}\)

\(^2\)It is helpful to make an association of **GOOD** or **BAD** with each qualifier, e.g., linear is **GOOD** and nonlinear is **BAD**.

\(^3\)Mathematicians regard **ordinary** derivatives as **plain and boring** whereas equations with 2\(^{nd}\) independent variables and **partial** derivatives are “**hot and spicy**”.

\(^4\)This textbook studies **ODEs** (ordinary differential equations), i.e., with one independent variable – not **PDEs**.
• First-order, second-order, third-order, \ldots, differential equations
  
The order of a differential equation is the order of the highest derivative of a dependent variable in the equation. If, after the farmer has put all dependent variables into the barn, there are any terms still out in the fields, the equation is inhomogeneous.

\begin{align*}
\sin(t) \left( \frac{dx}{dt} \right)^3 + \tan(x) &= 0 & \text{1st-order ODE} \\
\frac{d^2x}{dt^2} + \left( \frac{dx}{dt} \right)^3 + \tan(x) &= 0 & \text{2nd-order ODE} \\
\partial_s^3 y + \partial y / \partial s &= 0 & \text{3rd-order PDE}
\end{align*}

• Constant, periodic, and variable coefficient differential equations
  
A differential equation is classified by the coefficients of factors containing dependent variables. Coefficients are either constant or depend exclusively on independent variables, e.g.,\footnote{A physical system that is governed by a n\textsuperscript{th}-order differential equation is called an n\textsuperscript{th}-order system.} Periodic-coefficient ODEs are a special subset of variable-coefficient ODEs whose solution involves Floquet theory.

\begin{tabular}{ |c|c| }
\hline
\textbf{Constant-coefficient} & All the coefficients are constants. \\
\textbf{Periodic-coefficient} & All the coefficients are constants or periodic. \\
\textbf{Variable-coefficient} & One or more coefficients are neither constant nor periodic. \\
\hline
\end{tabular}

Complete each blank below with constant, periodic, or variable.

\begin{align*}
\dot{x} + 3x - \sin(t) &= 0 & \text{constant-coefficient, 2nd-order, ODE} \\
\dot{x} + tx &= 0 & \text{variable-coefficient, 2nd-order, ODE} \\
3\ddot{x} + x\dot{x} + 7\sin(x)x &= 0 & \text{constant-coefficient, 2nd-order, ODE} \\
\sin(t)\dot{x} + x &= 0 & \text{periodic-coefficient, 1st-order, ODE}
\end{align*}

2. Homogeneous and inhomogeneous (differential or algebraic equations)
  
A differential equation is classified by whether it not all its terms have a dependent variable.

\begin{tabular}{ |c|c| }
\hline
\textbf{Homogeneous} & All terms contain one or more dependent variables. \\
\textbf{Inhomogeneous} & One term is constant or depends exclusively on independent variables. \\
\hline
\end{tabular}

Complete each example below by circling homogeneous or inhomogeneous.

\begin{align*}
x + y &= 0 & \text{homogeneous/inhomogeneous algebraic equation} \\
x + y &= t & \text{homogeneous/inhomogeneous algebraic equation} \\
l^2 \ddot{x} + \sin(t) \dot{x}^2 + x &= 0 & \text{homogeneous/inhomogeneous ODE} \\
\dot{x} + 3 &= 0 & \text{homogeneous/inhomogeneous ODE}
\end{align*}

Note: Equations are frequently written in a “standard form” with all the unknowns (e.g., terms containing \( x, \dot{x}, \ddot{x}, y, \dot{y}, \ddot{y} \)) on the left-hand side of the equal sign and all the remaining terms on the right-hand side. The equation’s right-hand side will either be 0, a constant, or exclusively a function of independent variables. If the right-hand side is 0, the equation is homogeneous otherwise it is inhomogeneous.\footnote{Separating terms with a dependent variable from those without is analogous to a farmer separating sheep from goats. If the farmer only has sheep, the herd is homogeneous. If, after the farmer has put all dependent variables into the barn, there are any terms still out in the fields, the equation is inhomogeneous.}

3. Linear and nonlinear (differential or algebraic equations)
  
To determine if an expression is linear or nonlinear, one must first ask “Linear in what?” In the context of differential equations, a differential equation is linear if it is linear in its dependent variables and their derivatives (e.g., linear in \( x, \dot{x}, \ddot{x}, y, \dot{y}, \ddot{y} \)).

For example, a linear 2\textsuperscript{nd}-order ODE with one dependent variable \( x \) can be expressed as

\[ c_0 + c_1 x + c_2 \dot{x} + c_3 \ddot{x} = 0 \]

where \( c_i \) (i=0,1,2,3) are not functions of \( x, \dot{x}, \ddot{x}, \) (but may be functions of \( t \)).
A linear 1\textsuperscript{st}-order ODE with dependent variables \( x, y, z \) can be expressed as

\[
c_0 + c_1 \cdot x + c_2 \cdot \dot{x} + c_3 \cdot y + c_4 \cdot \dot{y} + c_5 \cdot z + c_6 \cdot \dot{z} = 0
\]

where \( c_i \) (\( i = 0, \ldots, 6 \)) are not functions of \( x, \dot{x}, y, \dot{y}, z, \dot{z} \), (but may be functions of \( t \)).

Complete each example below by circling linear or nonlinear.

\[
[t^2 + \sin(t)] \cdot x + \tan(t) = 4 \quad \text{linear/nonlinear} \quad \text{inhomogeneous algebraic equation}
\]

\[
x^2 + \sin(x) = 4 \quad \text{linear} \quad \text{inhomogeneous algebraic equation}
\]

\[
t^2 \ddot{x} + \sin(t) \dot{x} + x = 0 \quad \text{linear/nonlinear} \quad \text{inhomogeneous ODE}
\]

\[
\dot{x} + \sin(x) = 0 \quad \text{linear} \quad \text{inhomogeneous ODE}
\]

Note: Solving nonlinear equations is usually significantly more difficult than solving linear equations. Even determining the number of solutions to nonlinear equations can be difficult, e.g., a nonlinear equation may have no solution, 1 solution, 2 solutions, 17 solutions, or infinite solutions. For example, the solutions to \( x^2 + \sin(x) = 4 \) are \( x \approx 1.736 \) and \( x \approx 2.194 \), and can be obtained by a variety of methods, e.g., Newton-Rhapson or graphing.

4. Uncoupled and coupled (differential or algebraic equations)

To determine if a set of equations is coupled, one must first ask "Coupled in what?". A set of equations is coupled when the same unknown appears in more than one equation. A set of ODEs is coupled if a dependent variable or its derivative (e.g., \( x, \dot{x}, \ddot{x} \)) appear in more than one equation.

Complete each blank below with coupled or uncoupled.

\[
t^2 \cos(x^3) + \sin(t) \cdot x^2 + x = 0 \quad \text{uncoupled} \quad \text{nonlinear inhomogeneous algebraic}
\]

\[
y + t^3 \sin(y) - 9 = 0 \quad \text{uncoupled} \quad \text{nonlinear inhomogeneous algebraic}
\]

\[
x + y = 0 \quad \text{coupled} \quad \text{linear inhomogeneous algebraic}
\]

\[
x = 7
\]

\[
t^2 \ddot{x} + \sin(t) \cdot x^2 + x = 0 \quad \text{uncoupled} \quad \text{nonlinear inhomogeneous ODE}
\]

\[
\ddot{y} + t^3 \sin(y) - \tan(t) = 0
\]

\[
\dot{x} + \dot{y} = 0 \quad \text{coupled} \quad \text{linear homogeneous ODE}
\]

\[
\ddot{x} = 0
\]

3.6 Example of classification of differential equations

The equations governing the motion of the system described in Section 3.1 are

\[
\ddot{q}_A = \frac{2 \left[ 508.89 \sin(q_A) - \sin(q_B) \cos(q_B) \ddot{q}_A \dot{q}_B \right]}{-21.556 + \sin^2(q_B)}
\]

\[
\ddot{q}_B = -\sin(q_B) \cos(q_B) \dot{q}_A^2
\]

This set of equations is classified as (circle the appropriate qualifiers)

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<td>2\textsuperscript{nd}-order</td>
<td>Differential</td>
</tr>
</tbody>
</table>

It is worth noting that in general, an algebraic equation is good if it is linear. A differential equation is good if it is linear, homogeneous, constant-coefficient, first-order, and ordinary.\(^8\)

\(^8\)A good equation means that it is easy to manipulate and obtain a solution. A bad equation means that instead of enjoying time with your friends, you are struggling with difficult mathematics.