

### 3.4 Independent, dependent, and specified variables

To classify a differential equation, four types of mathematical quantities need to be identified.

- An **independent variable** is a quantity that varies independently, i.e., it is not controlled by (or depend on) other variables.<sup>1</sup> For many dynamic systems, there is one independent variable that is either  $t$  (time) or  $s$  (a complex variable used in conjunction with the Laplace transform).
- The symbol  $x$  is a **dependent variable** if its value depends on the independent variable and its dependence is considered to be **unknown**, e.g.,  $x$  is governed by a differential equation.
- The symbol  $x$  is a **specified variable** if its dependence on the independent variable can be considered to be **known**. For example,  $x$  is specified (or **prescribed**) when  $x = \sin(t)$  is **known** a priori.
- The symbol  $c$  is a **constant** if its value does not change, i.e., its value does not depend on the independent variable.

### 3.5 Classification of algebraic and differential equations

Uncoupled	Linear	Homogeneous	Constant-coefficient	1 <sup>st</sup> -order	Algebraic
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	2 <sup>nd</sup> -order	Differential

In the following examples,  $x$  and  $y$  are dependent variables and  $t$  is the independent variable time.<sup>2</sup>

#### 1. Differential and algebraic equations

In each example below, circle **differential** if the equation contains a time-derivative of a dependent variable (unknown), otherwise circle **algebraic**.

Note: Answers to these interactive questions are at [www.MotionGenesis.com](http://www.MotionGenesis.com) ⇒ [Textbooks](#) ⇒ [Resources](#).

- |   |   |
|---|---|
| $\frac{dy}{dt} = 7$   | <b>differential</b> /algebraic equation |
| $3 * y = 7$   | differential/ <b>algebraic</b> equation |
| $\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial s} = 0$ | <b>differential</b> /algebraic equation |
| $7y^4 + \cos(t) \sin(y)^3 = \tan(t) + 99$                               | differential/ <b>algebraic</b> equation |

- **Ordinary and partial differential equations**

When the dependent variable  $x$  is a function of only *one* independent variable  $t$ , the derivative of  $x$  with respect to  $t$  is called an **ordinary derivative**,<sup>3</sup> and is denoted  $\frac{dx}{dt}$  or  $\dot{x}$  or  $x'$ . Alternately, if  $x$  is a function of *two or more* independent variables (e.g.,  $s$  and  $t$ ), the derivative of  $x$  with respect to  $t$  is called the **partial derivative** of  $x$  with respect to  $t$ , and is denoted  $\frac{\partial x}{\partial t}$ .<sup>4</sup>

In each example below, circle **ordinary** if the equation is an **ODE**, i.e., contains only ordinary derivatives of dependent variables. Circle **partial** if the equation is an **PDE**, i.e., contains a partial derivative of a dependent variable.

- |   |  |
|---|--|
| $\frac{dx}{dt} = e^t$   | <b>ordinary</b> /partial differential equation |
| $\frac{\partial y}{\partial t} = 0$   | ordinary/ <b>partial</b> differential equation |
| $\frac{d^2 x}{dt^2} + \sin(t) \left(\frac{dx}{dt}\right)^2 + \tan(x) = e^t$ | <b>ordinary</b> /partial differential equation |
| $\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial s^2} = 0$ | ordinary/ <b>partial</b> differential equation |

<sup>1</sup>To help remember  $t$  is an **independent** variable, associate  $t$  with an **independent**, uncontrollable, [ ]teenager.

<sup>2</sup>It is helpful to make an association of **GOOD** or **BAD** with each qualifier, e.g., linear is **GOOD** and nonlinear is **BAD**.

<sup>3</sup>Mathematicians regard **ordinary** derivatives as **plain and boring** whereas equations with 2<sup>+</sup> independent variables and **partial** derivatives are **“hot and spicy”**.

<sup>4</sup>This textbook studies **ODEs** (ordinary differential equations), i.e., with **one** independent variable – not **PDEs**.

- **First-order, second-order, third-order, . . . , differential equations**

The order of a differential equation is the order of the highest derivative of a dependent variable in the equation.<sup>5</sup> Complete each blank below with 1<sup>st</sup>-order, 2<sup>nd</sup>-order, or 3<sup>rd</sup>-order.

$$\sin(t) \left(\frac{dx}{dt}\right)^3 + \tan(x) = 0 \quad \text{1}^{\text{st}}\text{-order ODE}$$

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 + \tan(x) = 0 \quad \text{2}^{\text{nd}}\text{-order ODE}$$

$$\frac{\partial^3 y}{\partial t^3} + \frac{\partial y}{\partial s} = 0 \quad \text{3}^{\text{rd}}\text{-order PDE}$$

- **Constant, periodic, and variable coefficient differential equations**

A differential equation is classified by the coefficients of factors containing dependent variables. Coefficients are either constant or depend exclusively on *independent variables*, e.g.,<sup>6</sup>

<b>Constant-coefficient</b>	<b>All</b> the coefficients are constants.
<b>Periodic-coefficient</b>	<b>All</b> the coefficients are constants or periodic.
<b>Variable-coefficient</b>	<b>One</b> or more coefficients are neither constant nor periodic.

Complete each blank below with **constant**, **periodic**, or **variable**.

$$\ddot{x} + 3 * x - \sin(t) = 0 \quad \text{constant}\text{-coefficient, } 2^{\text{nd}}\text{-order, ODE}$$

$$\ddot{x} + t * x = 0 \quad \text{variable}\text{-coefficient, } 2^{\text{nd}}\text{-order, ODE}$$

$$3\ddot{x} + x * \dot{x} + 7 * \sin(x) * x = 0 \quad \text{constant}\text{-coefficient, } 2^{\text{nd}}\text{-order, ODE}$$

$$\sin(t) * \dot{x} + x = 0 \quad \text{periodic}\text{-coefficient, } 1^{\text{st}}\text{-order, ODE}$$

## 2. Homogeneous and inhomogeneous (differential or algebraic equations)

A differential equation is classified by whether it not all its terms have a dependent variable.

<b>Homogeneous</b>	<b>All</b> terms contain one or more dependent variables.
<b>Inhomogeneous</b>	<b>One</b> term is constant or depends exclusively on <i>independent variables</i>

Complete each example below by circling **homogeneous** or **inhomogeneous**.

$$x + y = 0 \quad \text{homogeneous/inhomogeneous algebraic equation}$$

$$x + y = t \quad \text{homogeneous/inhomogeneous algebraic equation}$$

$$t^2 \ddot{x} + \sin(t) \dot{x}^2 + x = 0 \quad \text{homogeneous/inhomogeneous ODE}$$

$$\ddot{x} + 3 = 0 \quad \text{homogeneous/inhomogeneous ODE}$$

Note: Equations are frequently written in a “standard form” with all the **unknowns** (e.g., terms containing  $x$ ,  $\dot{x}$ ,  $\ddot{x}$ ,  $y$ ,  $\dot{y}$ ,  $\ddot{y}$ ) on the left-hand side of the equal sign and all the remaining terms on the right-hand side. The equation’s right-hand side will either be 0, a constant, or exclusively a function of independent variables. If the right-hand side is 0, the equation is *homogeneous* otherwise it is *inhomogeneous*.<sup>7</sup>

## 3. Linear and nonlinear (differential or algebraic equations)

To determine if an expression is *linear* or *nonlinear*, one must first ask “*Linear in what?*”. In the context of differential equations, a differential equation is *linear* if it is linear in its *dependent variables* and their derivatives (e.g., linear in  $x$ ,  $\dot{x}$ ,  $\ddot{x}$ ,  $y$ ,  $\dot{y}$ ,  $\ddot{y}$ ).

For example, a *linear* 2<sup>nd</sup>-order ODE with one dependent variable  $x$  can be expressed as

$$c_0 + c_1 * x + c_2 * \dot{x} + c_3 * \ddot{x} = 0$$

where  $c_i$  ( $i=0,1,2,3$ ) are **not** functions of  $x$ ,  $\dot{x}$ ,  $\ddot{x}$ , (but may be functions of  $t$ ).

<sup>5</sup>A physical system that is governed by a  $n^{\text{th}}$ -order differential equation is called an  $n^{\text{th}}$ -order system.

<sup>6</sup>Periodic-coefficient ODEs are a special subset of variable-coefficient ODEs whose solution involves Floquet theory.

<sup>7</sup>Separating terms with a dependent variable from those without is analogous to a farmer separating sheep from goats. If the farmer only has sheep, the herd is *homogeneous*. If, after the farmer has put all dependent variables into the barn, there are any terms still out in the fields, the equation is *inhomogeneous*.

A **linear** 1<sup>st</sup>-order ODE with dependent variables  $x, y, z$  can be expressed as

$$c_0 + c_1 * x + c_2 * \dot{x} + c_3 * y + c_4 * \dot{y} + c_5 * z + c_6 * \dot{z} = 0$$

where  $c_i$  ( $i = 0, \dots, 6$ ) are **not** functions of  $x, \dot{x}, y, \dot{y}, z, \dot{z}$ , (but may be functions of  $t$ ).

Complete each example below by circling **linear** or **nonlinear**.

$[t^2 + \sin(t)]x + \tan(t) = 4$	<b>linear</b> /nonlinear	inhomogeneous algebraic equation
$x^2 + \sin(x) = 4$	linear/ <b>nonlinear</b>	inhomogeneous algebraic equation
$t^2 \ddot{x} + \sin(t)\dot{x} + x = 0$	<b>linear</b> /nonlinear	homogeneous ODE
$\ddot{x} + \sin(x) = 0$	linear/ <b>nonlinear</b>	homogeneous ODE

Note: Solving **nonlinear equations** is usually significantly **more difficult** than solving **linear equations**. Even determining the **number** of solutions to nonlinear equations can be difficult, e.g., a nonlinear equation may have no solution, 1 solution, 2 solutions, 17 solutions, or infinite solutions. For example, the solutions to  $x^2 + \sin(x) = 4$  are  $x \approx 1.736$  and  $x \approx 2.194$ , and can be obtained by a variety of methods, e.g., Newton-Raphson or graphing.

#### 4. Uncoupled and coupled (differential or algebraic equations)

To determine if a set of equations is **coupled**, one must first ask “*Coupled in what?*”. A set of equations is **coupled** when the same unknown appears in more than one equation. A set of ODEs is coupled if a **dependent variable** or its derivative (e.g.,  $x, \dot{x}, \ddot{x}$ ) appear in more than one equation.

Complete each blank below with **coupled** or **uncoupled**.

$t^2 \cos(x^3) + \sin(t)x^2 + x = 0$	<b>uncoupled</b>	nonlinear inhomogeneous algebraic
$y + t^3 \sin(y) - 9 = 0$		

$x + y = 0$	<b>coupled</b>	linear inhomogeneous algebraic
$x = 7$		

$t^2 \ddot{x} + \sin(t)\dot{x}^2 + x = 0$	<b>uncoupled</b>	nonlinear inhomogeneous ODE
$\ddot{y} + t^3 \sin(y) - \tan(t) = 0$		

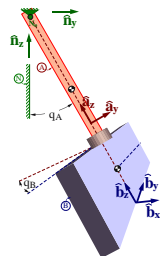
$\dot{x} + \dot{y} = 0$	<b>coupled</b>	linear homogeneous ODE
$\ddot{x} = 0$		

### 3.6 Example of classification of differential equations

The equations governing the motion of the system described in Section 3.1 are

$$\ddot{q}_A = \frac{2 [508.89 \sin(q_A) - \sin(q_B) \cos(q_B) \dot{q}_A \dot{q}_B]}{-21.556 + \sin^2(q_B)}$$

$$\ddot{q}_B = -\sin(q_B) \cos(q_B) \dot{q}_A^2$$



This set of equations is classified as (circle the appropriate qualifiers)

Uncoupled	Linear	<b>Homogeneous</b>	<b>Constant-coefficient</b>	1 <sup>st</sup> -order	Algebraic
<b>Coupled</b>	<b>Nonlinear</b>	Inhomogeneous	Variable-coefficient	<b>2<sup>nd</sup>-order</b>	<b>Differential</b>

It is worth noting that in general, an algebraic equation is **good** if it is linear. A differential equation is **good** if it is linear, homogeneous, constant-coefficient, first-order, and ordinary.<sup>8</sup>

<sup>8</sup>A **good** equation means that it is easy to manipulate and obtain a solution. A **bad** equation means that instead of enjoying time with your friends, you are struggling with difficult mathematics.