3.4 Independent, dependent, and specified variables

To classify a differential equation, four types of mathematical quantities need to be identified.

- An **_independent variable_** is a quantity that varies independently, i.e., it is not controlled by (or depend on) other variables.[1] For many dynamic systems, there is one independent variable that is either t (time) or s (a complex variable used in conjunction with the Laplace transform).

- The symbol x is a **_dependent variable_** if its value depends on the independent variable and its dependence is considered to be **unknown**, e.g., x is governed by a differential equation.

- The symbol x is a **_specified variable_** if its dependence on the independent variable can be considered to be **known**. For example, x is specified (or **_prescribed_**) when $x = sin(t)$ is **known** apriori.

- The symbol c is a **_constant_** if its value does not change, i.e., its value does not depend on the independent variable.

3.5 Classification of algebraic and differential equations

Uncoupled	Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

In the following examples, x and y are dependent variables and t is the independent variable time.[2]

1. **Differential and algebraic equations**
   In each example below, circle **_differential_** if the equation contains a time-derivative of a dependent variable (unknown), otherwise circle **_algebraic_**.

   Note: Answers to these interactive questions are at www.MotionGenesis.com $\Rightarrow$ Textbooks $\Rightarrow$ Resources.

$$\frac{dy}{dt} = 7 \qquad\qquad \textbf{differential/algebraic } \text{equation}$$

$$3 * y = 7 \qquad\qquad \textbf{differential/algebraic } \text{equation}$$

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial s} = 0 \qquad\qquad \textbf{differential/algebraic } \text{equation}$$

$$7y^4 + \cos(t)\sin(y)^3 = \tan(t) + 99 \qquad\qquad \textbf{differential/algebraic } \text{equation}$$

- **Ordinary and partial differential equations**
  When the dependent variable x is a function of only *one* independent variable t, the derivative of x with respect to t is called an **_ordinary derivative_**,[3] and is denoted $\frac{dx}{dt}$ or $\dot{x}$ or x'. Alternately, if x is a function of *two or more* independent variables (e.g., s and t), the derivative of x with respect to t is called the **_partial derivative_** of x with respect to t, and is denoted $\frac{\partial x}{\partial t}$.[4]
  In each example below, circle **_ordinary_** if the equation is an **_ODE_**, i.e., contains only ordinary derivatives of dependent variables. Circle **_partial_** is the equation is an **_PDE_**, i.e., contains a partial derivative of a dependent variable.

$$\frac{dx}{dt} = e^t \qquad\qquad \textbf{ordinary/partial } \text{differential equation}$$

$$\frac{\partial y}{\partial t} = 0 \qquad\qquad \textbf{ordinary/partial } \text{differential equation}$$

$$\frac{d^2 x}{dt^2} + \sin(t)\left(\frac{dx}{dt}\right)^2 + \tan(x) = e^t \qquad\qquad \textbf{ordinary/partial } \text{differential equation}$$

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial s^2} = 0 \qquad\qquad \textbf{ordinary/partial } \text{differential equation}$$

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[1]To help remember t is an **_independent_** variable, associate t with an **_independent_**, uncontrollable, []teenager.

[2]It is helpful to make an association of **GOOD** or **BAD** with each qualifier, e.g., linear is **GOOD** and nonlinear is **BAD**.

[3]Mathematicians regard **_ordinary_** derivatives as **_plain and boring_** whereas equations with 2^+ independent variables and **_partial_** derivatives are "**_hot and spicy_**" .

[4]This textbook studies **_ODEs_** (ordinary differential equations), i.e., with **one** independent variable – not **_PDEs_**.

- **First-order, second-order, third-order, ..., differential equations**
  The order of a differential equation is the order of the highest derivative of a dependent variable in the equation.[5]   Complete each blank below with 1st-**order**, 2nd-**order**, or 3rd-**order**.

$$\sin(t)\left(\frac{dx}{dt}\right)^3 + \tan(x) = 0$$ ⬜-order ODE

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 + \tan(x) = 0$$ ⬜-order ODE

$$\frac{\partial^3 y}{\partial t^3} + \frac{\partial y}{\partial s} = 0$$ ⬜-order PDE

- **Constant, periodic, and variable coefficient differential equations**
  A differential equation is classified by the coefficients of factors containing dependent variables. Coefficients are either constant or depend exclusively on ***independent variables***, e.g.,[6]

Constant-coefficient	***All*** the coefficients are constants.
Periodic-coefficient	***All*** the coefficients are constants or periodic.
Variable-coefficient	***One*** or more coefficients are neither constant nor periodic.

Complete each blank below with **constant**, **periodic**, or **variable**.

$$\ddot{x} + 3*x - \sin(t) = 0$$ ⬜-coefficient, 2nd-order, ODE

$$\ddot{x} + t*x = 0$$ ⬜-coefficient, 2nd-order, ODE

$$3\ddot{x} + x*\dot{x} + 7*\sin(x)*x = 0$$ ⬜-coefficient, 2nd-order, ODE

$$\sin(t)*\dot{x} + x = 0$$ ⬜-coefficient, 1st-order, ODE

2. **Homogeneous and inhomogeneous (differential or algebraic equations)**
   A differential equation is classified by whether it not all its terms have a dependent variable.

Homogeneous	***All*** terms contain one or more dependent variables.
Inhomogeneous	***One*** term is constant or depends exclusively on ***independent variables***

Complete each example below by circling **homogeneous** or **inhomogeneous**.

$$x + y = 0$$ **homogeneous/inhomogeneous** algebraic equation

$$x + y = t$$ **homogeneous/inhomogeneous** algebraic equation

$$t^2\ddot{x} + \sin(t)\dot{x}^2 + x = 0$$ **homogeneous/inhomogeneous** ODE

$$\ddot{x} + 3 = 0$$ **homogeneous/inhomogeneous** ODE

Note: Equations are frequently written in a "standard form" with all the **unknowns** (e.g., terms containing x, $\dot{x}$, $\ddot{x}$, y, $\dot{y}$, $\ddot{y}$) on the left-hand side of the equal sign and all the remaining terms on the right-hand side. The equation's right-hand side will either be 0, a constant, or exclusively a function of independent variables. If the right-hand side is 0, the equation is ***homogeneous*** otherwise it is ***inhomogeneous***.[7]

3. **Linear and nonlinear (differential or algebraic equations)**
   To determine if an expression is ***linear*** or ***nonlinear***, one must first ask *"Linear in what?"*. In the context of differential equations, a differential equation is ***linear*** if it is linear in its ***dependent variables*** and their derivatives (e.g., linear in x, $\dot{x}$, $\ddot{x}$, y, $\dot{y}$, $\ddot{y}$).

   For example, a ***linear*** 2nd-order ODE with one dependent variable x can be expressed as

$$c_0 + c_1*x + c_2*\dot{x} + c_3*\ddot{x} = 0$$

   where c_i (i=0,1,2,3) are **not** functions of x, $\dot{x}$, $\ddot{x}$, (but may be functions of t).

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[5]A physical system that is governed by a nth-order differential equation is called an nth-order system.

[6]Periodic-coefficient ODEs are a special subset of variable-coefficient ODEs whose solution involves Floquet theory.

[7]Separating terms with a dependent variable from those without is analogous to a farmer separating sheep from goats. If the farmer only has sheep, the herd is ***homogeneous***. If, after the farmer has put all dependent variables into the barn, there are any terms still out in the fields, the equation is ***inhomogeneous***.

A *linear* 1st-order ODE with dependent variables x, y, z can be expressed as

$$c_0 \ + \ c_1 * x \ + \ c_2 * \dot{x} \ + \ c_3 * y \ + \ c_4 * \dot{y} \ + \ c_5 * z \ + \ c_6 * \dot{z} \ = \ 0$$

where c_i (i = 0, …, 6) are **not** functions of x, $\dot{x}$, y, $\dot{y}$, z, $\dot{z}$, (but may be functions of t).

Complete each example below by circling **linear** or **nonlinear**.

$\left[t^2 + \sin(t)\right] x \ + \ \tan(t) \ = \ 4$	**linear/nonlinear** inhomogeneous algebraic equation
$x^2 \ + \ \sin(x) \ = \ 4$	**linear/nonlinear** inhomogeneous algebraic equation
$t^2 \ddot{x} \ + \ \sin(t)\dot{x} \ + \ x \ = \ 0$	**linear/nonlinear** homogeneous ODE
$\ddot{x} \ + \ \sin(x) \ = \ 0$	**linear/nonlinear** homogeneous ODE

Note: Solving *nonlinear equations* is usually significantly *more difficult* than solving *linear equations*. Even determining the *number* of solutions to nonlinear equations can be difficult, e.g., a nonlinear equation may have no solution, 1 solution, 2 solutions, 17 solutions, or infinite solutions. For example, the solutions to $x^2 + \sin(x) = 4$ are $x \approx 1.736$ and $x \approx 2.194$, and can be obtained by a variety of methods, e.g., Newton-Rhapson or graphing.

4. **Uncoupled and coupled (differential or algebraic equations)**
   To determine if a set of equations is *coupled*, one must first ask *"Coupled in what?"*. A *set* of equations is *coupled* when the same unknown appears in more than one equation. A set of ODEs is coupled if a *dependent variable* or its derivative (e.g., x, $\dot{x}$, $\ddot{x}$) appear in more than one equation.

   Complete each blank below with *coupled* or *uncoupled*.

   $$t^2 \cos(x^3) \ + \ \sin(t)\,x^2 \ + \ x \ = \ 0$$
   $$y \ + \ t^3 \sin(y) \ - \ 9 \ = \ 0$$

   [ ] nonlinear inhomogeneous algebraic

   $$x \ + \ y \ = \ 0$$
   $$x \ = \ 7$$

   [ ] linear inhomogeneous algebraic

   $$t^2\ddot{x} \ + \ \sin(t)\dot{x}^2 \ + \ x \ = \ 0$$
   $$\ddot{y} \ + \ t^3 \sin(y) \ - \ \tan(t) \ = \ 0$$

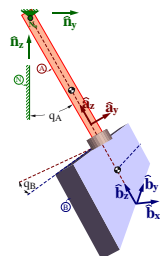   [ ] nonlinear inhomogeneous ODE

   $$\dot{x} \ + \ \dot{y} \ = \ 0$$
   $$\ddot{x} \ = \ 0$$

   [ ] linear homogeneous ODE

3.6   Example of classification of differential equations

The equations governing the motion of the system described in Section 3.1 are

$$\ddot{q}_A \ = \ \frac{2\left[508.89\,\sin(q_A) \ - \ \sin(q_B)\,\cos(q_B)\,\dot{q}_A\,\dot{q}_B\right]}{-21.556 + \sin^2(q_B)}$$

$$\ddot{q}_B \ = \ -\sin(q_B)\,\cos(q_B)\,\dot{q}_A^2$$

This set of equations is classified as (circle the appropriate qualifiers)

Uncoupled	Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

It is worth noting that in general, an algebraic equation is **good** if it is linear. A differential equation is **good** if it is linear, homogeneous, constant-coefficient, first-order, and ordinary.[8]

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[8] A **good** equation means that it is easy to manipulate and obtain a solution. A **bad** equation means that instead of enjoying time with your friends, you are struggling with difficult mathematics.