

## 6.7 Solution process for linear, $2^{nd}$ -order, ODEs

To solve equation (2)  $\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = 0$ , begin by assuming a solution of the form<sup>2</sup>

$$y(t) = C e^{pt} \quad \text{whose derivatives are} \quad \frac{dy}{dt} = p C e^{pt} \quad \text{and} \quad \frac{d^2y}{dt^2} = p^2 C e^{pt} \quad (9)$$

where  $C$  and  $p$  are **constants** to be determined.<sup>3</sup>

Substitute  $y(t)$ ,  $\dot{y}(t)$ ,  $\ddot{y}(t)$  from eqn(9) into (2).  $p^2 C e^{pt} + 2\zeta\omega_n (p C e^{pt}) + \omega_n^2 (C e^{pt}) = 0$  (2)

$$\text{Factor out } C e^{pt} \text{ to the right.} \quad [p^2 + 2\zeta\omega_n p + \omega_n^2] * C * e^{pt} = 0$$

Section 5.4 and equation (5.6) show  $e^{pt} \neq 0$ .  $[p^2 + 2\zeta\omega_n p + \omega_n^2] * C = 0$

Use  $C \neq 0$ . "**Characteristic equation**":  $[p^2 + 2\zeta\omega_n p + \omega_n^2] = 0$

Solve for  $p$  (the values of  $p$  are called "**poles**").  $p_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$  (10)

Assemble the solution.  $y(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t}$

### 6.7.1 Solution process for linear, $n^{th}$ -order, constant-coefficient ODEs

The **characteristic equation** in the previous section was a **quadratic equation** for  $p$  and had **two** roots,  $p_1$  and  $p_2$ . More generally, the solution process in the previous section works for **any order** linear, constant-coefficient, ODE and results in a **characteristic polynomial equation** in  $p$ .

|                 |   |   |
|-----------------|---|---|
| $1^{st}$ -order | $a_1 p + a_0 = 0$                                       | Solve linear equation as $p = -a_0/a_1$ .               |
| $2^{nd}$ -order | $a_2 p^2 + a_1 p + a_0 = 0$                             | Solve with <b>quadratic equation</b> .                  |
| $3^{rd}$ -order | $a_3 p^3 + a_2 p^2 + a_1 p + a_0 = 0$                   | Solve with calculator or computer.                      |
| $n^{th}$ -order | $a_n p^n + a_{n-1} p^{n-1} + a_2 p^2 + a_1 p + a_0 = 0$ | Solve with calculator or computer (eigenvalue routine). |

### 6.7.2 Derivation of solution for **undamped**, second-order, ODE

When  $\zeta = 0$  undamped, equation (2) simplifies to  $\frac{d^2y}{dt^2} + \omega_n^2 y = 0$  and the roots  $p_1$  and  $p_2$  simplify to

$$p_1 = +i\omega_n \quad p_2 = -i\omega_n \quad (11)$$

With these two roots and Euler's formula [equation (5.2)], there are two solutions of the ODE.

$$y_1(t) = C_1 e^{p_1 t} = C_1 e^{i\omega_n t} = C_1 [\cos(\omega_n t) + i \sin(\omega_n t)] \quad (12)$$

$$y_2(t) = C_2 e^{p_2 t} = C_2 e^{-i\omega_n t} = C_2 [\cos(\omega_n t) - i \sin(\omega_n t)] \quad (13)$$

$$y(t) = y_1(t) + y_2(t) = (C_1 + C_2) \cos(\omega_n t) + (C_1 - C_2) i \sin(\omega_n t) \quad (14)$$

Since  $C_1$  and  $C_2$  are yet-to-be-determined constants, they can be replaced by two new other undetermined constants  $A$  and  $B$  (defined below) so  $y(t)$  can be written as:

$$A \triangleq C_1 + C_2 \quad B \triangleq (C_1 - C_2) i \quad \Rightarrow \quad y(t) = A \cos(\omega_n t) + B \sin(\omega_n t) \quad (14)$$

<sup>2</sup>Assume a solution of the form  $C e^{pt}$  because it worked for first-order ODEs and it is the best guess we have.

<sup>3</sup>Note: This proof mimics Section 4.3. Answers are at [www.MotionGenesis.com](http://www.MotionGenesis.com)  $\Rightarrow$  [Textbooks](#)  $\Rightarrow$  [Resources](#).  
Note:  $C \neq 0$  since  $C = 0$  gives the trivial (not good) zero solution  $y(t) = 0$ .