

## 5.7 Solution process for linear, 2<sup>nd</sup>-order, ODEs

To solve equation (2)  $\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = 0$ , begin by assuming a solution of the form (assume a solution of the form  $C e^{pt}$  because it worked for 1<sup>st</sup>-order ODEs in Section 4.3 and it is the best guess we have).

$$y(t) = C e^{pt} \quad \text{whose 1}^{st} \text{ and 2}^{nd} \text{ derivatives are} \quad \frac{dy}{dt} = \quad \frac{d^2y}{dt^2} = \quad (10)$$

where  $C$  and  $p$  are **constants** to be determined [ $C \neq 0$  since  $C = 0$  gives the trivial (boring) solution  $y(t) = 0$ ].

Substitute  $y(t)$ ,  $\dot{y}(t)$ ,  $\ddot{y}(t)$  from eqn (10) into (2).  $\quad + 2\zeta\omega_n \left( \quad \right) + \omega_n^2 \left( \quad \right) = 0$  (2)

Factor out  $C e^{pt}$  to the right.  $\left[ \quad \right] * C * e^{pt} = 0$

Section 7.1 and equation (7.1) show  $e^{pt} \neq 0$ .  $\left[ \quad \right] * C = 0$

Use  $C \neq 0$  to get “**characteristic equation**”:  $\left[ \quad \right] = 0$

Solve for  $p$  (the values of  $p$  are called “**poles**”).  $p_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$  (11)

Assemble the solution.  $y(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t}$

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### 5.7.1 Solution process for linear, n<sup>th</sup>-order, constant-coefficient ODEs

The **characteristic equation** in the previous section was a **quadratic equation** for  $p$  and had **two** roots,  $p_1$  and  $p_2$ . More generally, the solution process in the previous section works for **any order** linear, constant-coefficient, ODE and results in a **characteristic polynomial equation** in  $p$ .

1 <sup>st</sup> -order	$a_1 p + a_0 = 0$	Solve linear equation as $p = -a_0/a_1$ .
2 <sup>nd</sup> -order	$a_2 p^2 + a_1 p + a_0 = 0$	Solve with <b>quadratic equation</b> .
3 <sup>rd</sup> -order	$a_3 p^3 + a_2 p^2 + a_1 p + a_0 = 0$	Solve with calculator or computer.
n <sup>th</sup> -order	$a_n p^n + \dots + a_3 p^3 + a_2 p^2 + a_1 p + a_0 = 0$	Solve with calculator or computer (eigenvalue routine).

### 5.7.2 Derivation of solution for undamped, second-order, ODE

When  $\zeta = 0$  undamped, equation (2) simplifies to  $\frac{d^2y}{dt^2} + \omega_n^2 y = 0$  and the roots  $p_1$  and  $p_2$  simplify to

$$p_1 \stackrel{(11)}{=} +i\omega_n \quad p_2 \stackrel{(11)}{=} -i\omega_n \quad (12)$$

With these two roots and Euler’s formula [equation (5.6)], there are two solutions of the ODE.

$$y_1(t) \stackrel{(10)}{=} C_1 e^{p_1 t} \stackrel{(12)}{=} C_1 e^{i\omega_n t} \stackrel{(5.6)}{=} C_1 [\cos(\omega_n t) + i \sin(\omega_n t)] \quad (13)$$

$$y_2(t) \stackrel{(10)}{=} C_2 e^{p_2 t} \stackrel{(12)}{=} C_2 e^{-i\omega_n t} \stackrel{(5.6)}{=} C_2 [\cos(\omega_n t) - i \sin(\omega_n t)] \quad (14)$$

$$y(t) = y_1(t) + y_2(t) \stackrel{(13,14)}{=} (C_1 + C_2) \cos(\omega_n t) + (C_1 - C_2) i \sin(\omega_n t) \quad (15)$$

Since  $C_1$  and  $C_2$  are yet-to-be-determined constants, they can be replaced by two new other undetermined constants  $A$  and  $B$  (defined below) so  $y(t)$  can be written as:

$$A \triangleq C_1 + C_2 \quad B \triangleq (C_1 - C_2) i \quad \Rightarrow \quad y(t) \stackrel{(15)}{=} A \cos(\omega_n t) + B \sin(\omega_n t)$$

Alternately,  $y(t)$  can be rewritten using the “**amplitude-phase**” formula [equation (2.21)] as

$$y(t) = \bar{C} \cos(\omega_n t + \phi) \quad \text{where} \quad \bar{C} = \sqrt{A^2 + B^2} \quad \phi = \text{atan2}(-A, B)$$