Chapter 25

Undamped coupled 2^{nd}-order ODEs

Summary (see examples in Hw 13)

Many physical phenomena are governed by a set of n coupled ***undamped*** linear, constant-coefficient, homogeneous ODEs. These include free vibrations of buildings, airplanes, automobiles, space structures, and molecules.

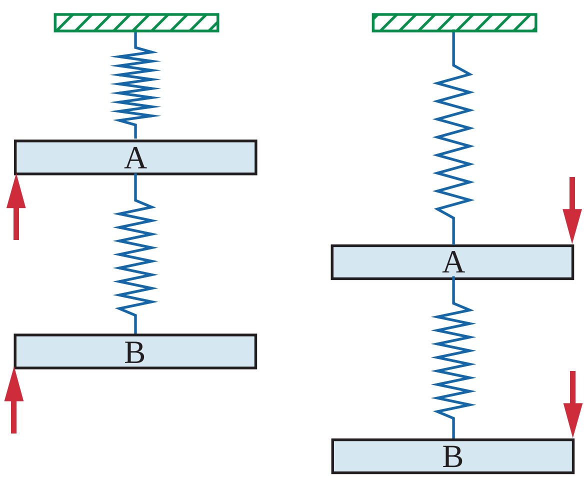
$$M\,\ddot{X} \;+ \overset{0}{\cancel{B}}\,\dot{X} \;+\; K\,X \;=\; [0] \qquad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

If $B = [0]$, the ODEs are ***undamped***.

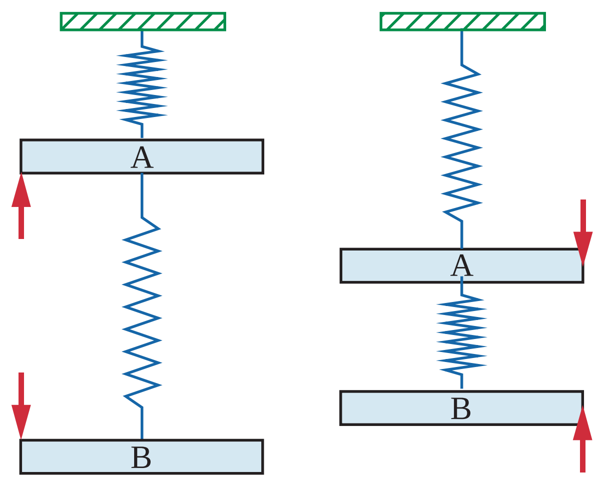
25.1 Physical insights into eigenvalues and eigenvectors of a slinky

Two rulers connected to a slinky (spring) demonstrate the physical significance of eigenvalues/eigenvectors.

Shown below are rulers A and B moving in the ***first mode***, moving up ↑ together or down ↓ together.



Shown below are rulers A and B moving in the ***second mode***, moving in opposite directions.



Mode #	Eigenvalue (frequency)	Eigenvector (use + or 0 or −)	Potential Energy ($\frac{1}{2}\,k\,\delta^2$)	Kinetic Energy ($\frac{1}{2}\,m\,v^2$)	Total Energy Kinetic + Potential
1	small/large	$\begin{bmatrix} + \\ + \end{bmatrix}$ or $\begin{bmatrix} - \\ - \end{bmatrix}$	small/large one/two deformed spring min/max deformation	small/large	small/large
2	small/large	$\begin{bmatrix} + \\ - \end{bmatrix}$ or $\begin{bmatrix} - \\ + \end{bmatrix}$	small/large one/two deformed springs min/max deformation	small/large	small/large

Answers at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

This motivating example shows that eigenvalues and eigenvectors are physically identifiable through how the rulers move (frequency and direction). As is apparent in this physical demonstration and is generally true, the total energy in a high frequency mode is larger than the energy in a low frequency mode.

25.2 Solution of *undamped, coupled* ODEs

$X(t)$	$n \times 1$ matrix of time-dependent variables
M	$n \times n$ matrix of constants (***mass matrix***)
B	$n \times n$ matrix of constants (***damping matrix***)
K	$n \times n$ matrix of constants (***stiffness matrix***)
G	$n \times p$ matrix of constants
$F(t)$	$p \times 1$ matrix of **known** functions of time $f_1(t) \ldots f_p(t)$ (***forcing function matrix***)

Any set of n coupled, linear, constant-coefficient, ***2^{nd}-order ODEs*** can be written in the matrix form

$$\underset{(n \times n)}{M} \ddot{X} + \underset{(n \times n)}{B} \dot{X} + \underset{(n \times n)}{K} X = \underset{(n \times p)}{G} \underset{(p \times 1)}{F} \qquad (1)$$

When $B = [0]$, this set of ODEs are ***undamped***.
When $B \neq [0]$, this set of ODEs are ***damped***.

To solve equation (1), write $X(t) = X_h(t) + X_p(t)$ (the sum of a homogeneous and particular solution). The homogeneous solution $X_h(t)$ can be found by assuming a solution of the form shown in equation (2), where p is a yet-to-be-determined constant and U is a yet-to-be-determined **non-zero** $n \times 1$ matrix of constants.[a]

$$X_h(t) = U e^{p\,t} \qquad (2)$$

$$U \triangleq \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix}$$

[a] It is reasonable to guess $X_h(t) = U e^{p\,t}$ because it worked for **uncoupled** linear ODEs. The matrix U is **non-zero** because $U = [0]$ produces the ***trivial solution*** $X_h(t) = [0]$, which is not a solution of interest (not what we are looking for).

Substituting/differentiating equation (2) into the homogeneous part of equation (1) $(F = [0])$, gives

$$M \left[p^2 U e^{p\,t} \right] + B \left[p\, U e^{p\,t} \right] + K \left[U e^{p\,t} \right] \underset{(1,2)}{=} [0] \qquad \text{Factor on } U e^{p\,t}$$

$$\left[p^2 M + p\,B + K \right] U e^{p\,t} = [0] \qquad \text{Section 7.1 shows } e^{p\,t} \neq 0$$

$$\left[p^2 M + p\,\overset{0}{\cancel{B}} + K \right] U = [0] \qquad \text{Sometimes } B \approx [0] \text{ (see **undamped** below)}$$

$$\left[p^2 M + K \right] U = [0] \qquad \text{Define: } \lambda \triangleq -p^2 \quad p = \pm\sqrt{-\lambda}$$

$$\textit{Generalized eigen-equation} \quad \left[-\lambda M + K \right] U = [0] \quad \Rightarrow \quad \det\left[-\lambda M + K \right] = 0 \qquad (3)$$

This eigen-equation is a **coupled** nonlinear algebraic equation. It has $\boldsymbol{n}$ equations and $\boldsymbol{n+1}$ unknowns in λ and U.
The related $\det[-\lambda M + K] = 0$ is **1** **uncoupled** nonlinear algebraic equation with **1** unknown λ.
Equation (1) ⇒ (3) changes $\boldsymbol{n}$ coupled ODEs with $\boldsymbol{n}$ unknowns into a ***eigen-equation***.

Undamped: It can be reasonable to set $B = [0]$ if:
- Damping is small.
- It is difficult to analytically or experimentally determine the elements of the damping matrix.
- In structural vibrations, damping is approximated with ***modal damping*** (see Section 25.6).

For eigen-equation (3) to produce a **non-zero** U (and **non-zero** X_h), the inverse of $[-\lambda M + K]$ must not exist. To see this, suppose $[-\lambda M + K]^{-1}$ does exist and multiply both sides of the eigen-equation by $[-\lambda M + K]^{-1}$, which produces $U = [0]$. For $U \neq [0]$, one must find ***special*** values of λ so $[-\lambda M + K]^{-1}$ does not exist. These ***special*** values $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called ***eigenvalues***. For each λ_i, there is a corresponding ***special*** non-zero U_i called the ***eigenvector*** corresponding to λ_i.
Note: The eigenvalue problem is a ***special*** nonlinear algebraic equation because the number of solutions is known apriori.

The following are equivalent statements about equation (3) and finding the ***eigenvalues*** λ:
- Find the values of λ which result in $U \neq [0]$.
- Find the values of λ so the matrix $[-\lambda M + K]$ is singular, i.e., $[-\lambda M + K]^{-1}$ does not exist.
- Find the values of λ so the determinant of $[-\lambda M + K]$ is zero, i.e., $\det[-\lambda M + K] = 0$