Chapter 23

Undamped coupled 2^{nd}-order ODEs

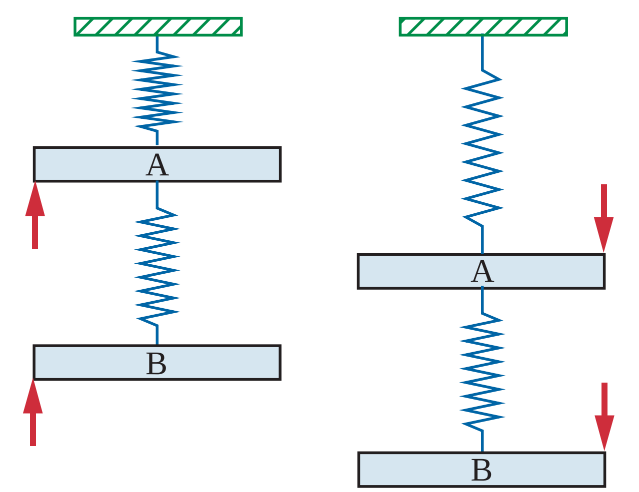
Summary (see examples in Hw 13)

Many physical phenomena are governed by **undamped**, coupled, linear, constant-coefficient, ODEs. These include certain motions of buildings, airplanes, automobiles, beams, space structures, and molecules.

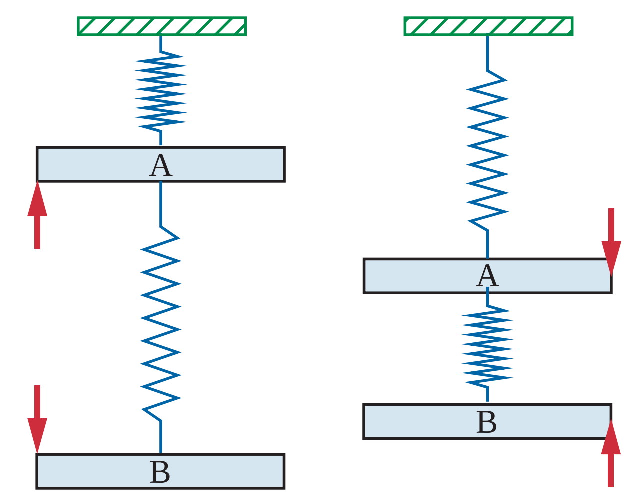
23.1 Physical insights into eigenvalues and eigenvectors of a slinky

Two long thin rulers (A and B) connected to a slinky (spring) are useful for demonstrating the physical significance of eigenvalues and eigenvectors.

Shown below is A and B moving in the ***first mode***, moving up ↑ together or down ↓ together.



Shown below is A and B moving in the ***second mode*** (moving in opposite directions).



Mode #	Eigenvalue (frequency)	Eigenvector (use + or 0 or −)	Potential Energy ($\frac{1}{2}\,k\,\delta^2$)	Kinetic Energy ($\frac{1}{2}\,m\,v^2$)	Total Energy Kinetic + Potential
1	**small/large**	$\begin{bmatrix}+\\ \square\end{bmatrix}$ or $\begin{bmatrix}-\\ \square\end{bmatrix}$	$\boxed{\textbf{small}}$**/large** $\boxed{\textbf{one}}$**/two** deformed spring $\boxed{\textbf{min}}$**/max** deformation	**small/large**	**small/large**
2	**small/large**	$\begin{bmatrix}+\\ \square\end{bmatrix}$ or $\begin{bmatrix}-\\ \square\end{bmatrix}$	**small/**$\boxed{\textbf{large}}$ **one/**$\boxed{\textbf{two}}$ deformed springs **min/**$\boxed{\textbf{max}}$ deformation	**small/large**	**small/large**

Answers at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

23.2 Analytical solutions of *undamped, coupled*, ODEs

Any set of n coupled, linear, constant-coefficient, ***2^{nd}-order ODEs*** may be written in the matrix form

$$\underset{(n\times n)}{M}\;\ddot{X} \;+\; \underset{(n\times n)}{B}\;\dot{X} \;+\; \underset{(n\times n)}{K}\;X \;=\; \underset{(n\times p)}{G}\;\underset{(p\times 1)}{F} \qquad (1)$$

where X is a $n\times 1$ matrix of dependent variables, M, B, K are $n\times n$ matrices of constants, G is a $n\times p$ matrix of constants, and F is a $p\times 1$ matrix of functions of time $f_1(t)$, $f_2(t)$, ..., $f_p(t)$.

To solve equation (1), write $X(t) = X_h(t) + X_p(t)$ (the sum of a homogeneous and particular solution). The homogeneous solution $X_h(t)$ can be found by assuming a solution of the form shown in equation (2), where p is a yet-to-be-determined constant and U is a yet-to-be-determined **non-zero** $n\times 1$ matrix of constants.[a]

$$X_h(t) \;=\; \boxed{U\,e^{p\,t}} \qquad (2)$$

$$U \;\triangleq\; \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix}$$

[a] It is reasonable to guess $X_h(t) = U\,e^{p\,t}$ because it worked for **uncoupled** linear ODEs. The matrix U is **non-zero** because $U = [0]$ produces the ***trivial solution*** $X_h(t) = [0]$, which is not a solution of interest (not what we are looking for).

Substituting/differentiating equation (2) into equation (1), factoring on $U e^{p\,t}$, and simplifying, gives

Note: This proof mimics Section 21.1. Answers are at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

$$M\Big(\underset{(2)}{\boxed{}}\Big) + B\Big(\underset{(2)}{\boxed{}}\Big) + K\Big(\underset{(2)}{\boxed{}}\Big) \;\underset{(1)}{=}\; [0]$$

$$\Big(\boxed{}\Big)\,U\,e^{p\,t} \;=\; [0] \qquad \text{Section 7.1 shows: } e^{p\,t} \neq 0$$

$$\Big(\boxed{}\Big)\,U \;=\; [0] \qquad \text{Sometimes } B \approx [0] \text{ (see below)}$$

$$\Big(\boxed{}\Big)\,U \;=\; [0] \qquad \text{Define: } \boxed{\lambda \triangleq -p^2} \quad \boxed{p = \pm\sqrt{-\lambda}}$$

$$\boxed{\textbf{\textit{Generalized eigenvalue equation}} \qquad (-\lambda\,M + K)\,U = [0] \;\;\Rightarrow\;\; \det(-\lambda\,M + K) = 0} \qquad (3)$$

Equation (3a) is a **coupled** nonlinear algebraic equation. Its two unknowns are λ and U.

Equation (3b) is an **uncoupled** nonlinear algebraic equation. Its only unknown is λ.

(3a) ⇒ (3b) changes ***n*** equations with ***n+1*** unknowns into **1** equation with **1** unknown.

Equation (1) ⇒ (3) changes ***n*** coupled ODEs with ***n*** unknowns into a ***generalized eigenvalue equation***.

Note: It is reasonable to set $B = [0]$ if damping is small. Other circumstances in which B is set to $[0]$:

- It is difficult to analytically or experimentally determine the elements of the damping matrix.
- It makes the mathematics easier.
- In structural vibrations, damping is approximated with ***modal damping***.

For equation (3a) to produce a **non-zero** U (and **non-zero** X_h), the inverse of $(-\lambda M+K)$ must not exist. To see this, suppose $(-\lambda M+K)^{-1}$ does exist and multiply both sides of equation (3a) by $(-\lambda M+K)^{-1}$ which produces $U = [0]$. To get a **non-zero** U from equation (3a), one must find values of λ so $(-\lambda M+K)^{-1}$ does not exist. These special values of λ are denoted λ_1, λ_2, ..., λ_n and are called ***eigenvalues***.

For each λ_i, there is a corresponding non-zero U_i called the ***eigenvector*** corresponding to λ_i.[1]

The following are equivalent statements about equation (3a) and finding the ***eigenvalues*** λ:

- Find the values of λ which result in $U \neq [0]$.
- Find the values of λ so the matrix $[-\lambda\,M + K]$ is singular, i.e., $[-\lambda\,M + K]^{-1}$ does not exist.
- Find the values of λ so the determinant of $[-\lambda\,M + K]$ is zero, i.e., $\boxed{\det[-\lambda\,M + K] = 0}$ (4)

[1] The eigenvalue problem is a special type of nonlinear algebraic equation because the number of solutions is known apriori.