Chapter 20

Eigenvalues and eigenvectors

$$\boxed{\begin{aligned} A\,U &= \lambda\,U \\ A\,U &= \lambda\,B\,U \end{aligned}}$$

What is a matrix eigenvalue problem? (see examples in Hw 12)

An ***eigenvalue*** is a "special value" of λ that allows equation (1) to produce ***non-zero*** U.[1]

- λ is an unknown ***scalar*** (called an ***eigenvalue***)
- $U \neq [\,0\,]$ is a unknown $n\times 1$ column matrix (called an ***eigenvector***)
- Matrix(λ) is an $n\times n$ matrix that depends on λ

$$\boxed{\text{Matrix}(\lambda) \;*\; U \;=\; [\,0\,]} \qquad (1)$$

$[\,0\,]$ is the $n\times 1$ zero matrix.

Eigenvalue problem	Equation form	Alternate form	Solution for λ
Standard eigenvalue	$[-\lambda\,I \,+\, A]\;U \;=\; [\,0\,]$	$A\,U \;=\; \lambda\,U$	$\det\,[-\lambda\,I \,+\, A] = 0$
Generalized eigenvalue	$[-\lambda\,B \,+\, A]\;U \;=\; [\,0\,]$	$A\,U \;=\; \lambda\,B\,U$	$\det\,[-\lambda\,B \,+\, A] = 0$
Quadratic eigenvalue	$[M\,\lambda^2 \,+\, B\,p + K]\;U \;=\; [\,0\,]$	Not applicable	$\det\,[M\,\lambda^2 \,+\, B\,p + K] = 0$
Nonlinear eigenvalue	Matrix(λ) $*$ $U \;=\; [\,0\,]$	Not applicable	$\det\,[\text{Matrix}(\lambda)] = 0$

20.1 Recognize and remember: Solving an eigenvalue problem

There are similarities between the familiar ***quadratic equation*** and an ***eigenvalue problem***. Both are algebraic equations that are ***nonlinear*** in their unknowns, and both have known solutions. It is important to ***recognize*** these equations and ***remember*** their solutions.

	Quadratic equation	**Standard eigenvalue**	**Generalized eigenvalue**
Equation form Alternate form	$a\,x^2 \,+\, b\,x \,+\, c \,=\, 0$ $a\,x^2 \,+\, b\,x \,=\, -c$	$(-\lambda\,I \,+\, A)\,U \;=\; [\,0\,]$ $A\,U \;=\; \lambda\,U$	$(-\lambda\,B \,+\, A)\,U \;=\; [\,0\,]$ $A\,U \;=\; \lambda\,B\,U$
Unknowns Equation type	x **Nonlinear**	$\lambda,\; U$ **Nonlinear**	$\lambda,\; U$ **Nonlinear**
Solution	$x \;=\; \dfrac{-b \,\pm\, \sqrt{b^2 \,-\, 4\,a\,c}}{2\,a}$	$\det\,(-\lambda\,I \,+\, A) \;=\; 0$	$\det\,(-\lambda\,B \,+\, A) \;=\; 0$

20.2 Motivating questions for eigenvalues and eigenvectors

Question 1: Consider the following equation that has two unknowns, namely λ and U. Since this equation has the special form $A\,U = \lambda\,U$ (with $A = 3$), it is ***recognized*** as an ***eigenvalue problem***.

$$(-\lambda \,+\, 3) \,*\, U \;=\; 0$$

The solution to this equation is a ***"special value"*** of λ and associated ***non-zero*** U,

Eigenvalue: $\lambda \;=\; \boxed{3}$ ***Eigenvector:*** $U \;=\; \boxed{\text{any number}}$

[1] U is a "right" eigenvector for Matrix(λ) $*\,U \;=\; [\,0\,]$, whereas U is a "left" eigenvector for $U *$ Matrix(λ) $=\; [\,0\,]$.

Question 2: Find a ***non-zero*** solution $y(t)$ to the ODE shown below-right.
Start by substituting the assumed solution $y(t) = U\, e^{p\,t}$ into the ODE where p is a constant (to-be-determined) and U is a ***non-zero*** constant.[a] Subsequently, rearrange and simplify using $e^{p\,t} \neq 0$.

The equation for p is ***recognized*** as an ***eigenvalue problem***.
The ***"special value"*** of p and associated ***non-zero*** U are[b]

ODE:	$\dot{y} - 3\,y$	$= 0$
Eigen-problem:	$(p - 3)\; U$	$= 0$
Solution:	$y(t)$	$= c\, e^{3\,t}$

Eigenvalue: $p = \boxed{3}$ ***Eigenvector:*** $U = \boxed{\text{any constant } c}$

[a] Note: $U = 0$ produces the trivial (degenerate) solution $y(t) = 0$, – which is not what we are looking for. Hence $U \neq 0$.
[b] Note: In ODEs, this ***"special value"*** of p is called a ***pole*** whereas in matrix algebra, p is called an ***eigenvalue***.

Question 3: Solve the following set of linear algebraic equations for x and y (for given values of d).

$$\begin{aligned} x \;-\; y &= 0 \\ x \;+\; d\,y &= 0 \end{aligned} \quad \overset{\Longleftrightarrow}{\text{or}} \quad \begin{bmatrix} 1 & -1 \\ 1 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Answers at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

$d = 0$	$x = \boxed{0}$	$y = \boxed{0}$
$d = 1$	$x = \boxed{0}$	$y = \boxed{0}$
$d = 2$	$x = \boxed{0}$	$y = \boxed{0}$
$d = 3$	$x = \boxed{0}$	$y = \boxed{0}$
$d = -1$	$x = \boxed{\text{any number}}$	$y = \boxed{x}$

Note: The ***special value*** $d = -1$ is the **only** value of d that produces a **non-zero** solution for x and y.
Note: One way to solve for this ***special value*** of d is by setting the determinant of the 2×2 matrix equal to 0.

Question 4: Eigenvalue and eigenvector concepts. (Answers: www.MotionGenesis.com ⇒ Textbooks ⇒ Resources)
Consider the following set of algebraic equations governing the unknowns u_1, u_2, and λ.

$$\begin{aligned} \lambda\, u_1 \;+\; u_2 &= 0 \\ 4\, u_1 \;+\; \lambda\, u_2 &= 0 \end{aligned} \qquad \text{or equivalently} \qquad \begin{bmatrix} \lambda & 1 \\ 4 & \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find "special values" of λ (called ***eigenvalues***) that allow for **non-zero** u_1 and u_2.
Result: $\lambda_1 = \boxed{2}$ $\qquad \lambda_2 = \boxed{-2}$

For each special value of λ determine a corresponding "special ratio" of u_2 to u_1.
Result: (These "special ratios" are called ***eigenvectors*** and c_1 and c_2 are arbitrary constants.)

$$\text{For } \lambda_1\text{:} \quad U_1 \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ \boxed{-2} \end{bmatrix} \qquad \text{For } \lambda_2\text{:} \quad U_2 \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ \boxed{2} \end{bmatrix}$$

Question 5: Eigenvalues for a non-standard, non-generalized eigenvalue problem
Consider the following set of algebraic equations governing the unknowns u_1, u_2, and λ.

$$\begin{aligned} \lambda^2\, u_1 \;+\; 5\, u_2 &= 0 \\ \big(\cos(\lambda) - 0.9\big)\, u_1 \;+\; \lambda\, u_2 &= 0 \end{aligned} \qquad \text{or equivalently} \qquad \begin{bmatrix} \boxed{\lambda^2} & \boxed{5} \\ \boxed{\cos(\lambda) - 0.9} & \boxed{\lambda} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find an equation, which when solved produces "special values" of λ that allow for **non-zero** u_1 and u_2.
Result: (These "special values" of λ are called ***eigenvalues***.)

$$\boxed{\boxed{\lambda^3} \;-\; \boxed{5\,\cos(\lambda)} \;+\; 4.5} \;=\; 0$$

Answers at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

Note: It is questionable whether this eigen-problem can be cast as a standard or generalized eigenvalue problem.
Three eigenvalues that satisfy this equation are: $\lambda_1 = -1.7574$, $\lambda_2 = -0.5078$, $\lambda_3 = +0.4166$.