

# Chapter 20

## Eigenvalues and eigenvectors

$$AU = \lambda U$$

$$AU = \lambda BU$$

What is a matrix eigenvalue problem? (see examples in Hw 12)

An *eigenvalue* is a “special value” of  $\lambda$  that allows equation (1) to produce non-zero  $U$ .<sup>1</sup>

- $\lambda$  is an unknown *scalar* (called an *eigenvalue*)
- $U \neq [0]$  is a unknown  $n \times 1$  column matrix (called an *eigenvector*)
- $\text{Matrix}(\lambda)$  is an  $n \times n$  matrix that depends on  $\lambda$

$$\text{Matrix}(\lambda) * U = [0] \quad (1)$$

$[0]$  is the  $n \times 1$  zero matrix.

Eigenvalue problem	Equation form	Alternate form	Solution for $\lambda$
<i>Standard eigenvalue</i>	$[-\lambda I + A] U = [0]$	$AU = \lambda U$	$\det[-\lambda I + A] = 0$
<i>Generalized eigenvalue</i>	$[-\lambda B + A] U = [0]$	$AU = \lambda BU$	$\det[-\lambda B + A] = 0$
<i>Quadratic eigenvalue</i>	$[M\lambda^2 + B\lambda + K] U = [0]$	Not applicable	$\det[M\lambda^2 + B\lambda + K] = 0$
<i>Nonlinear eigenvalue</i>	$\text{Matrix}(\lambda) * U = [0]$	Not applicable	$\det[\text{Matrix}(\lambda)] = 0$

### 20.1 Recognize and remember: Solving an eigenvalue problem

There are similarities between the familiar *quadratic equation* and an *eigenvalue problem*. Both are algebraic equations that are nonlinear in their unknowns, and both have known solutions. It is important to recognize these equations and remember their solutions.

	Quadratic equation	Standard eigenvalue	Generalized eigenvalue
Equation form	$ax^2 + bx + c = 0$	$(-\lambda I + A)U = [0]$	$(-\lambda B + A)U = [0]$
Alternate form	$ax^2 + bx = -c$	$AU = \lambda U$	$AU = \lambda BU$
Unknowns	$x$	$\lambda, U$	$\lambda, U$
Equation type	<b>Nonlinear</b>	<b>Nonlinear</b>	<b>Nonlinear</b>
Solution	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$\det(-\lambda I + A) = 0$	$\det(-\lambda B + A) = 0$

### 20.2 Motivating questions for eigenvalues and eigenvectors

**Question 1:** Consider the following equation that has two unknowns, namely  $\lambda$  and  $U$ . Since this equation has the special form  $AU = \lambda U$  (with  $A = 3$ ), it is recognized as an *eigenvalue problem*.

$$(-\lambda + 3) * U = 0$$

The solution to this equation is a “*special value*” of  $\lambda$  and associated non-zero  $U$ ,

**Eigenvalue:**  $\lambda = 3$                       **Eigenvector:**  $U =$  any number

<sup>1</sup> $U$  is a “right” eigenvector for  $\text{Matrix}(\lambda) * U = [0]$ , whereas  $U$  is a “left” eigenvector for  $U * \text{Matrix}(\lambda) = [0]$ .

**Question 2:** Find a **non-zero** solution  $y(t)$  to the ODE shown below-right.

Start by substituting the assumed solution  $y(t) = U e^{pt}$  into the ODE where  $p$  is a constant (to-be-determined) and  $U$  is a **non-zero** constant.<sup>a</sup> Subsequently, rearrange and simplify using  $e^{pt} \neq 0$ .

The equation for  $p$  is **recognized** as an **eigenvalue problem**.

The “**special value**” of  $p$  and associated **non-zero**  $U$  are<sup>b</sup>

**Eigenvalue:**  $p = 3$       **Eigenvector:**  $U =$  any constant  $c$

ODE:	$\dot{y} - 3y = 0$
Eigen-problem:	$(p - 3)U = 0$
Solution:	$y(t) = c e^{3t}$

<sup>a</sup>Note:  $U = 0$  produces the trivial (degenerate) solution  $y(t) = 0$ , - which is **not** what we are looking for. Hence  $U \neq 0$ .

<sup>b</sup>Note: In ODEs, this “**special value**” of  $p$  is called a **pole** whereas in matrix algebra,  $p$  is called an **eigenvalue**.

**Question 3:** Solve the following set of linear algebraic equations for  $x$  and  $y$  (for given values of  $d$ ).

$$\begin{array}{l} x - y = 0 \\ x + dy = 0 \end{array} \quad \begin{array}{c} \iff \\ \text{or} \end{array} \quad \begin{bmatrix} 1 & -1 \\ 1 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$d = 0$	$x = 0$	$y = 0$
$d = 1$	$x = 0$	$y = 0$
$d = 2$	$x = 0$	$y = 0$
$d = 3$	$x = 0$	$y = 0$
$d = -1$	$x =$ any number	$y = x$

Answers at [www.MotionGenesis.com](http://www.MotionGenesis.com)  $\Rightarrow$  [Textbooks](#)  $\Rightarrow$  [Resources](#).

Note: The **special value**  $d = -1$  is the **only** value of  $d$  that produces a **non-zero** solution for  $x$  and  $y$ .

Note: One way to solve for this **special value** of  $d$  is by setting the determinant of the  $2 \times 2$  matrix equal to 0.

**Question 4: Eigenvalue and eigenvector concepts.** (Answers: [www.MotionGenesis.com](http://www.MotionGenesis.com)  $\Rightarrow$  [Textbooks](#)  $\Rightarrow$  [Resources](#))

Consider the following set of algebraic equations governing the unknowns  $u_1$ ,  $u_2$ , and  $\lambda$ .

$$\begin{array}{l} \lambda u_1 + u_2 = 0 \\ 4 u_1 + \lambda u_2 = 0 \end{array} \quad \text{or equivalently} \quad \begin{bmatrix} \lambda & 1 \\ 4 & \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find “special values” of  $\lambda$  (called **eigenvalues**) that allow for **non-zero**  $u_1$  and  $u_2$ .

**Result:**  $\lambda_1 = 2$        $\lambda_2 = -2$

For each special value of  $\lambda$  determine a corresponding “special ratio” of  $u_2$  to  $u_1$ .

**Result:** (These “special ratios” are called **eigenvectors** and  $c_1$  and  $c_2$  are arbitrary constants.)

$$\text{For } \lambda_1: U_1 \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{For } \lambda_2: U_2 \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

**Question 5: Eigenvalues for a non-standard, non-generalized eigenvalue problem**

Consider the following set of algebraic equations governing the unknowns  $u_1$ ,  $u_2$ , and  $\lambda$ .

$$\begin{array}{l} \lambda^2 u_1 + 5 u_2 = 0 \\ (\cos(\lambda) - 0.9) u_1 + \lambda u_2 = 0 \end{array} \quad \text{whose matrix form is:} \quad \begin{bmatrix} \lambda^2 & 5 \\ \cos(\lambda) - 0.9 & \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find an equation, which when solved produces “special values” of  $\lambda$  that allow for **non-zero**  $u_1$  and  $u_2$ .

**Result:** (These “special values” of  $\lambda$  are called **eigenvalues**.)  $\lambda^3 - 5 \cos(\lambda) + 4.5 = 0$

Note: It is questionable whether this eigen-problem can be cast as a standard or generalized eigenvalue problem.

Three eigenvalues that satisfy this equation are:  $\lambda_1 = -1.7574$ ,  $\lambda_2 = -0.5078$ ,  $\lambda_3 = +0.4166$ .