Chapter 22

Eigenvalues and eigenvectors

$$\boxed{\begin{aligned} AU &= \lambda U \\ AU &= \lambda BU \end{aligned}}$$

What is a matrix eigenvalue problem?   (see examples in Hw 12)

An **_eigenvalue_** is a "special value" of λ that allows equation (1) to produce **_non-zero_** U.[1]

- λ is an unknown **_scalar_** (called an **_eigenvalue_**).
- $U \neq [0]$ is a unknown $n \times 1$ column matrix (called an **_eigenvector_**).
- $\text{Matrix}(\lambda)$ is an $n \times n$ matrix that depends on λ.

$$\boxed{\text{Matrix}(\lambda) \; * \; U \; = \; [0]} \qquad (1)$$

$[0]$ is the $n \times 1$ zero matrix.

Eigenvalue problem	Equation form	Alternative form	Solution for λ
Standard eigenvalue	$[-\lambda I + A]\,U \; = \; [0]$	$AU \; = \; \lambda U$	$\det[-\lambda I + A] = 0$
Generalized eigenvalue	$[-\lambda B + A]\,U \; = \; [0]$	$AU \; = \; \lambda BU$	$\det[-\lambda B + A] = 0$
Quadratic eigenvalue	$[M\lambda^2 + B\lambda + K]\,U \; = \; [0]$	Not applicable	$\det[M\lambda^2 + B\lambda + K] = 0$
Nonlinear eigenvalue	$\text{Matrix}(\lambda) \; * \; U \; = \; [0]$	Not applicable	$\det[\text{Matrix}(\lambda)] = 0$

22.1   Recognize and remember: Solving an eigenvalue problem

There are similarities between the familiar **_quadratic equation_** and an **_eigenvalue problem_**. Both are algebraic equations that are **_nonlinear_** in their unknowns, and both have known solutions. It is important to **_recognize_** these equations and **_remember_** their solutions.

	Quadratic equation	**Standard eigenvalue**	**Generalized eigenvalue**
Equation form	$a\,x^2 + b\,x + c = 0$	$[-\lambda I + A]\,U \; = \; [0]$	$[-\lambda B + A]\,U \; = \; [0]$
Alternative form	$a\,x^2 + b\,x \; = \; -c$	$AU \; = \; \lambda U$	$AU \; = \; \lambda BU$
Unknowns	x	$\lambda,\; U$	$\lambda,\; U$
Equation type	**Nonlinear**	**Nonlinear**	**Nonlinear**
Solution	$x = \dfrac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$	$\det[-\lambda I + A] \; = \; 0$	$\det[-\lambda B + A] \; = \; 0$

22.2   Motivating questions for eigenvalues and eigenvectors   (Hw 12.9)

Question 1:   Solve the following equation for the two unknowns λ and U (with the condition $U \neq 0$). Since this equation has the special form $\text{Matrix}(\lambda)\,U = 0$, it is **_recognized_** as an **_eigenvalue problem_**.

$$[-\lambda + 3] \; * \; U \; = \; 0 \qquad (\text{or alternatively} \;\; 3\,U = \lambda\,U\,)$$

The solution to this equation is a **_"special value"_** of λ and an associated **_non-zero_** U.

Eigenvalue: $\lambda = \boxed{3}$ **_Eigenvector:_** $U = \boxed{\text{any number}}$

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[1] U is a "right" eigenvector for $\text{Matrix}(\lambda) * U = [0]$, whereas U is a "left" eigenvector for $U * \text{Matrix}(\lambda) = [0]$.

Question 2: Find a **_non-zero_** solution $y(t)$ to the ODE shown below-right.

Start by substituting the assumed solution $y(t) = U e^{pt}$ into the ODE where p is a constant (to-be-determined) and U is a **_non-zero_** constant.[a] Subsequently, rearrange and simplify using $e^{pt} \neq 0$.

The equation for p is **_recognized_** as an **_eigenvalue problem_**.
The **"special value"** of p and associated **_non-zero_** U are[b]

$$
\begin{aligned}
\text{ODE:} && \dot{y} - 3y &= 0 \\
\text{Eigen-problem:} && (p - 3)\,U &= 0 \\
\text{Solution:} && y(t) &= U\,e^{3t}
\end{aligned}
$$

   Eigenvalue: $p = \boxed{3}$ **_Eigenvector:_** $U = \boxed{\text{any constant}}$

[a] Note: $U = 0$ produces the trivial (degenerate) solution $y(t) = 0$, which is <u>not</u> what we are looking for. Hence $U \neq 0$.
[b] Note: In ODEs, this **"special value"** of p is called a **_pole_** whereas in matrix algebra p is called an **_eigenvalue_**.

Question 3: Eigenvalue and eigenvector concepts. (Answers: <u>www.MotionGenesis.com</u> ⇒ <u>Textbooks</u> ⇒ <u>Resources</u>)

Consider the following set of algebraic equations governing the unknowns u_1, u_2, and λ.

$$
\begin{aligned}
\lambda u_1 + u_2 &= 0 \\
4 u_1 + \lambda u_2 &= 0
\end{aligned}
\qquad \underset{\text{or}}{\Longleftrightarrow} \qquad
\begin{bmatrix} \lambda & 1 \\ 4 & \lambda \end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Find "special values" of λ (called **_eigenvalues_**) that allow for $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Result:
$$ \lambda_1 = \boxed{2} \qquad\qquad \lambda_2 = \boxed{} $$

For each special value of λ determine a corresponding "special ratio" of u_2 to u_1.
Result: (These "special ratios" are called **_eigenvectors_** and c_1 and c_2 are arbitrary constants.)

$$
\text{For } \lambda_1: \quad U_1 \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ \boxed{-2} \end{bmatrix}
\qquad\qquad
\text{For } \lambda_2: \quad U_2 \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ \boxed{} \end{bmatrix}
$$

Question 4: Eigenvalues for an unusual (nonlinear) eigenvalue problem.

Consider the following set of algebraic equations governing the unknowns u_1, u_2, and λ.

$$
\begin{aligned}
\lambda^2 u_1 + 5 u_2 &= 0 \\
\big(\cos(\lambda) - 0.9\big) u_1 + \lambda u_2 &= 0
\end{aligned}
\quad
\begin{array}{c} \text{whose matrix} \\ \text{form is:} \end{array}
\quad
\begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Find an equation, which when solved produces "special values" of λ that allow for $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Result: (These "special values" of λ are called **_eigenvalues_**.) $\boxed{} = 0$

Note: It is questionable whether this eigen-problem can be cast as a standard or generalized eigenvalue problem.
Three eigenvalues that satisfy this equation are: $\lambda_1 = -1.7574, \quad \lambda_2 = -0.5078, \quad \lambda_3 = +0.4166$.

Question 5: Solve the following set of linear algebraic equations for x and y (for given values of d).

$$
\begin{aligned}
x - y &= 0 \\
x + d\,y &= 0
\end{aligned}
\quad \underset{\text{or}}{\Longleftrightarrow} \quad
\begin{bmatrix} 1 & -1 \\ 1 & d \end{bmatrix}
\begin{bmatrix} x \\ y \end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

$d = 0$	$x = \boxed{0}$	$y = \boxed{0}$
$d = 1$	$x = \boxed{}$	$y = \boxed{}$
$d = 2$	$x = \boxed{}$	$y = \boxed{}$
$d = 3$	$x = \boxed{0}$	$y = \boxed{0}$
$d = -1$	$x = \boxed{}$	$y = \boxed{}$

Answers at <u>www.MotionGenesis.com</u> ⇒ <u>Textbooks</u> ⇒ <u>Resources</u>.

Note: The **_special value_** $d = -1$ is the **only** value of d that produces a **non-zero** solution for x and y.
Note: One way to solve for this **_special value_** of d is by setting the determinant of the 2×2 matrix equal to 0.
Note: This is recognized as an eigenvalue problem **if** the question is: Find the special value of d that allows for $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.