Chapter 15

Complex numbers

$$i \triangleq \sqrt{-1}$$
$$e^{i\,\theta} \;=\; \cos(\theta) \,+\, i\,\sin(\theta)$$

Summary: Tools and use for complex numbers (see examples in Hw 8, 9)

This chapter provides algebraic tools for complex numbers ($+$, $*$, $/$, $\sqrt{\ }$, and powers).
Complex numbers are useful in many aspects of dynamic systems, including:

Solving 2^{nd}, 3^{rd}, and higher-order ODEs	Root locus	Circuit analysis
Control system design	Frequency response	Eigen-analysis

Motivating the imaginary number i

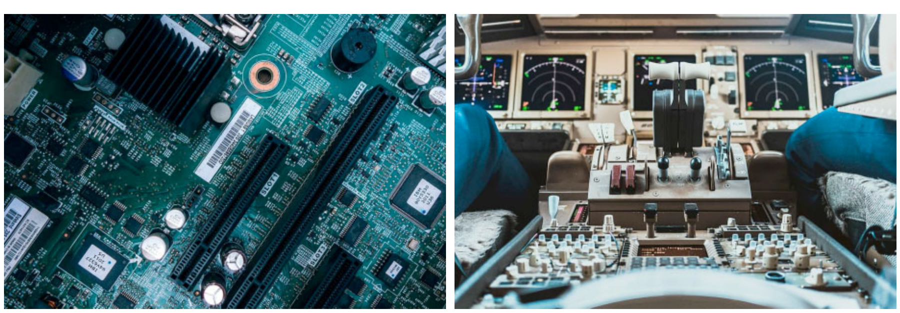
- The following invalid proof that $1 = -1$ involves the imaginary number i defined as $i \triangleq \sqrt{-1}$. Circle the incorrect step in the proof and explain your reasoning (solution in footnote at bottom of page).

$$1 \;=\; \sqrt{1} \;=\; \boxed{\sqrt{(-1)^2} \;=\; \sqrt{-1} * \sqrt{-1}} \;=\; i * i \;=\; i^2 \;=\; -1$$

- Find all real and/or complex numbers that can appear on the right-hand side of the equal signs.

$$1^4 \;=\; \left(1\,e^{2n\pi\,i}\right)^4_{\;n=0,1,2,\ldots} \;=\; e^{8n\pi\,i} \;=\; \cos(8n\pi) + i\,\sin(8n\pi) \;=\; \boxed{1}$$
$$1^{1/4} \;=\; \left(1\,e^{2n\pi\,i}\right)^{1/4}_{\;n=0,1,2,\ldots} \;=\; e^{\frac{n\,\pi}{2}\,i} \;=\; \cos(\tfrac{n\,\pi}{2}) + i\,\sin(\tfrac{n\,\pi}{2}) \;=\; \boxed{\pm 1,\; \pm i}$$
$$1^{1/3} \;=\; \left(1\,e^{2n\pi\,i}\right)^{1/3}_{\;n=0,1,2,\ldots} \;=\; e^{\frac{2n\pi}{3}\,i} \;=\; \cos(\tfrac{2n\pi}{3}) + i\,\sin(\tfrac{2n\pi}{3}) \;=\; \boxed{1,\; -0.5 \pm 0.866\,i}$$

If needed: Answers to these interactive questions are at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.



Complex numbers are used in circuit analysis and control system design.

[0]The incorrect step is $\sqrt{(-1)^2} \neq \sqrt{-1} * \sqrt{-1}$. When x is negative, $(x^2)^{\frac{1}{2}} \neq (x^{\frac{1}{2}})^2$. In general, $\boxed{(x^a)^b \;\neq\; (x^b)^a}$.

$$1^4 \;=\; \left(1\,e^{2n\pi\,i}\right)^4_{\;n=0,1,2,\ldots} \;=\; e^{8n\pi\,i} \;=\; \cos(8n\pi) + i\,\sin(8n\pi) \;=\; \boxed{1}$$
$$1^{1/4} \;=\; \left(1\,e^{2n\pi\,i}\right)^{1/4}_{\;n=0,1,2,\ldots} \;=\; e^{\frac{n\,\pi}{2}\,i} \;=\; \cos(\tfrac{n\,\pi}{2}) + i\,\sin(\tfrac{n\,\pi}{2}) \;=\; \boxed{\pm 1,\; \pm i} \quad \textbf{Or}\;\; 1^{1/4} \;=\; \sqrt{\sqrt{1}} \;=\; \sqrt{\pm 1} \;=\; \pm 1 \text{ or } \pm i$$
$$1^{1/3} \;=\; \left(1\,e^{2n\pi\,i}\right)^{1/3}_{\;n=0,1,2,\ldots} \;=\; e^{\frac{2n\pi}{3}\,i} \;=\; \cos(\tfrac{2n\pi}{3}) + i\,\sin(\tfrac{2n\pi}{3}) \;=\; \boxed{1,\; -0.5 \pm 0.866\,i}$$