

# Chapter 15

## Complex numbers

$$i \triangleq \sqrt{-1}$$

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

**Summary: Tools and use for complex numbers** (see examples in Hw 8, 9)

This chapter provides algebraic tools for complex numbers (+, \*, /,  $\sqrt{\phantom{x}}$ , and powers).  
Complex numbers are useful in many aspects of dynamic systems, including:

Solving 2 <sup>nd</sup> , 3 <sup>rd</sup> , and higher-order ODEs	Root locus	Circuit analysis
Control system design	Frequency response	Eigen-analysis

### Motivating the imaginary number $i$

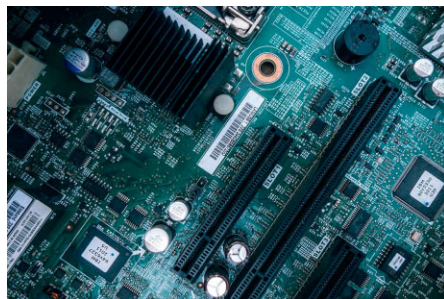
- The following invalid proof that  $1 = -1$  involves the imaginary number  $i$  defined as  $i \triangleq \sqrt{-1}$ .  
Circle the incorrect step in the proof and explain your reasoning (solution in footnote at bottom of page).

$$1 = \sqrt{1} = \sqrt{(-1)^2} = \sqrt{-1} * \sqrt{-1} = i * i = i^2 = -1$$

- Find all real and/or complex numbers that can appear on the right-hand side of the equal signs.

$$\begin{aligned} 1^4 &= \text{[yellow box]} = \text{[yellow box]} = \text{[yellow box]} = \boxed{1} \\ 1^{1/4} &= \text{[yellow box]} = \text{[yellow box]} = \text{[yellow box]} = \boxed{\pm 1, \pm i} \\ 1^{1/3} &= \text{[yellow box]} = \text{[yellow box]} = \text{[yellow box]} = \boxed{1, -0.5 \pm 0.866i} \end{aligned}$$

If needed: Answers to these interactive questions are at [www.MotionGenesis.com](http://www.MotionGenesis.com)  $\Rightarrow$  [Textbooks](#)  $\Rightarrow$  [Resources](#).



Complex numbers are used in circuit analysis and control system design.

<sup>0</sup>The incorrect step is  $\sqrt{(-1)^2} \neq \sqrt{-1} * \sqrt{-1}$ . When  $x$  is negative,  $(x^2)^{\frac{1}{2}} \neq (x^{\frac{1}{2}})^2$ . In general,  $(x^a)^b \neq (x^b)^a$ .

$$\begin{aligned} 1^4 &= (1 e^{2n\pi i})^4 \quad n=0, 1, 2, \dots = e^{8n\pi i} = \cos(8n\pi) + i \sin(8n\pi) = \boxed{1} \\ 1^{1/4} &= (1 e^{2n\pi i})^{1/4} \quad n=0, 1, 2, \dots = e^{\frac{n\pi}{2} i} = \cos(\frac{n\pi}{2}) + i \sin(\frac{n\pi}{2}) = \boxed{\pm 1, \pm i} \quad \text{Or } 1^{1/4} = \sqrt[4]{1} = \sqrt{\pm 1} = \pm 1 \text{ or } \pm i \\ 1^{1/3} &= (1 e^{2n\pi i})^{1/3} \quad n=0, 1, 2, \dots = e^{\frac{2n\pi}{3} i} = \cos(\frac{2n\pi}{3}) + i \sin(\frac{2n\pi}{3}) = \boxed{1, -0.5 \pm 0.866i} \end{aligned}$$