

# Chapter 15

## Complex numbers

$$i \triangleq \sqrt{-1}$$
$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

**Summary: Tools and use for complex numbers** (see examples in Hw 8, 9)

This chapter provides algebraic tools for complex numbers (+, \*, /,  $\sqrt{\phantom{x}}$ , and powers).<sup>1</sup>  
Complex numbers are useful in many aspects of dynamic systems, including:

Solving 2 <sup>nd</sup> , 3 <sup>rd</sup> , and higher-order ODEs	Root locus	
Control system design	Frequency response	Eigen-analysis

### Motivating questions

- The following proof involves the imaginary number  $i$  defined as  $i \triangleq \sqrt{-1}$  and shows the surprising conclusion that  $1 = -1$ . Circle the incorrect step in the proof and explain your reasoning.

$$1 = \sqrt{1} = \sqrt{(-1)^2} = \sqrt{-1} * \sqrt{-1} = i * i = i^2 = -1$$



- Find all real and/or complex numbers that can appear on the right-hand side of the equal signs.

$$1^4 = \text{[yellow box]} = \text{[yellow box]} = \text{[yellow box]} = \text{[1]}$$
$$1^{1/4} = \text{[yellow box]} = \text{[yellow box]} = \text{[yellow box]} = \text{[±1, ±i]}$$
$$1^{1/3} = \text{[yellow box]} = \text{[yellow box]} = \text{[yellow box]} = \text{[1, -0.5 ± 0.866i]}$$

Answers at [www.MotionGenesis.com](http://www.MotionGenesis.com) ⇒ [Textbooks](#) ⇒ [Resources](#). Alternate:  $1^{1/4} = \sqrt{\sqrt{1}} = \sqrt{\pm 1} = \pm 1$  or  $\pm i$ .

<sup>1</sup>Complex numbers are said to be “**closed**” under addition, subtraction, negation, multiplication, division, and exponentiation because when these operations are performed on complex numbers, only complex numbers result. Complex numbers are said to be “**algebraically closed**” because polynomial equations with complex number coefficients can only produce complex numbers. Real numbers are not closed under exponentiation. For example,  $-4^{0.5} = \sqrt{-4} = \pm 2i$  produces a complex (not real) number. Real numbers are not algebraically closed. For example, although the polynomial equation  $x^2 + 2x + 5 = 0$  has real coefficients, its roots are the complex numbers  $x = -1 \pm 2i$ .