# Chapter 13

# Complex numbers

$$i \triangleq \sqrt{-1}$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

## Summary: Tools and use for complex numbers (see examples in Hw 8, 9)

This chapter provides algebraic tools for complex numbers ($+$, $*$, $/$, $\sqrt{\ }$, and powers).[1]
Complex numbers are useful in many aspects of dynamic systems, including:

| | | |
|---|---|---|
| Solving $2^{nd}$, $3^{rd}$, and higher-order ODEs | Root locus | |
| Control system design | Frequency response | Eigen-analysis |

## Motivating questions

- The following proof involves the imaginary number $i$ defined as $i \triangleq \sqrt{-1}$ and shows the surprising conclusion that $1 = -1$. Circle the incorrect step in the proof and explain your reasoning.

$$1 \;=\; \sqrt{1} \;=\; \sqrt{(-1)^2} \;=\; \sqrt{-1} * \sqrt{-1} \;=\; i * i \;=\; i^2 \;=\; -1$$

- Find all real and/or complex numbers that can appear on the right-hand side of the equal signs.[2]

$1^4 =$ [ ] $=$ [ ] $=$ [ ] $=$ [$1$]

$1^{1/4} =$ [ ] $=$ [ ] $=$ [ ] $=$ [$\pm 1,\ \pm i$]

$1^{1/3} =$ [ ] $=$ [ ] $=$ [ ] $=$ [$1,\ -0.5 \pm 0.866\,i$]

[1] Complex numbers are said to be "***closed***" under addition, subtraction, negation, multiplication, division, and exponentiation because when these operations are performed on complex numbers, only complex numbers result. Complex numbers are said to be "***algebraically closed***" because polynomial equations with complex number coefficients can only produce complex numbers. Real numbers are not closed under exponentiation. For example, $-4^{0.5} = \sqrt{-4} = \pm 2\,i$ produces a complex (not real) number. Real numbers are not algebraically closed. For example, although the polynomial equation $x^2 + 2\,x + 5 = 0$. has real coefficients, its roots are the complex numbers $x = -1 \pm 2\,i$.

[2] Answers at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources. Alternate: $1^{1/4} = \sqrt{\sqrt{1}} = \sqrt{\pm 1} = \pm 1$ or $\pm i$.