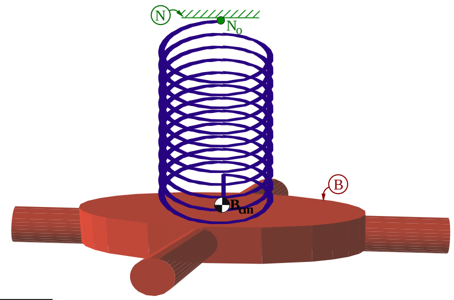
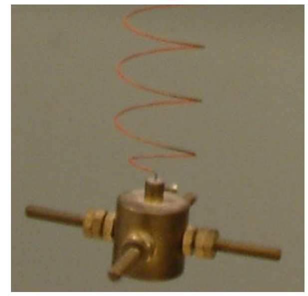
13.6 Coupled motions of WilberForce pendulum



A rigid body B is attached to a spring at B_{cm} (B's center of mass). The spring's other end is attached to point N_o fixed in a Newtonian reference frame N. This system's motion is governed by

$$m\,\ddot{x} \;+\; k_x\,x \;+\; k_c\,\theta \;=\; 0$$
$$I\,\ddot{\theta} \;+\; k_c\,x \;+\; k_\theta\,\theta \;=\; 0$$

Quantity	Symbol	Type
B's mass	m	Positive constant
B's central moment of inertia about vertical axis	I	Positive constant
Extensional linear spring constant modeling	k_x	Positive constant
Torsional linear spring constant modeling	k_θ	Positive constant
Coupled linear spring constant modeling	k_c	Positive constant
Translational stretch of spring from equilibrium	x	Variable
Rotational stretch of spring from equilibrium	θ	Variable



Purchase Wilburforce pendulum at PASCO Scientific or B&B Co. (Super Spinnerama).
More WilberForce information at "WilberForce pendulum oscillations and normal modes", Berg and Marshall, Am. J. Physics Vol. 59, No. 1, January 1991.

(a) Write the ODEs in the matrix form $M\,\ddot{X} + B\,\dot{X} + K\,X \;=\; [0]$. Next, substitute the assumed solution $\boxed{X(t) = U\,e^{p\,t}}$ where p is a constant (to-be-determined) and U is a ***non-zero*** 2×1 matrix of constants (to-be-determined). Show p and U are governed by a ***generalized eigenvalue problem***. Next, define $\lambda \triangleq -p^2$ and find a <u>**scalar**</u> equation that governs λ.

Result:

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}}_{\dot{X}} + \underbrace{\begin{bmatrix} k_x & k_c \\ k_c & k_\theta \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[p^2\,M \;+\; K\right]\,U\,e^{p\,t} \;=\; [0] \qquad\Rightarrow\qquad \left[p^2\,M \;+\; K\right]\,U \;=\; [0]$$

$$\boxed{\lambda \;\triangleq\; -p^2} \quad\Rightarrow\quad \left[-\lambda\;M + K\right]\,U \;=\; [0] \qquad\Rightarrow\qquad \det\left[-\lambda\,M + K\right] \;=\; 0$$

(b) Show this can be recast as a ***standard eigenvalue problem*** $[-\lambda\,I + A]\,U \;=\; [0]$.
Result – determine A: (note I is the 2×2 identity matrix)

$$[-\lambda\,I + A]\,U \;=\; [0] \qquad \text{where} \qquad A \;=\; M^{-1}\,K \;=\; \begin{bmatrix} k_x/m & k_c/m \\ k_c/I & k_\theta/I \end{bmatrix}$$

(c) Complete the two blanks in the following polynomial equation that governs $\boxed{\lambda \;\triangleq\; -p^2}$.
Result: (Note: The blanks only involve m, I, k_x, k_θ, k_c).

$$\lambda^2 \;+\; \left(\frac{-k_x}{m} + \frac{-k_\theta}{I}\right) * \lambda \;+\; \frac{k_x\,k_\theta - k_c^2}{m\,I} \;=\; 0$$

(d) For certain values of m, I, k_x, k_θ, k_c, the matrix $A \;\triangleq\; M^{-1}\,K \;=\; \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$.
Find the ***eigenvalues*** and corresponding ***eigenvectors*** of A.
Result:

$\lambda_1 = 9$ $\qquad \lambda_2 = 11$

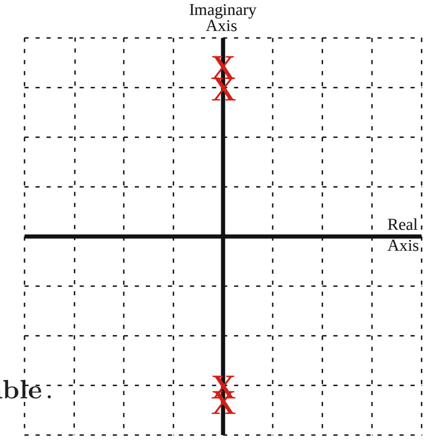
$U_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\qquad U_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

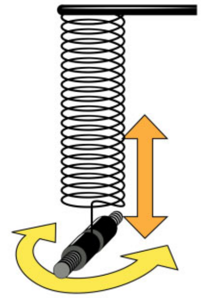
Calculate the values of p_1, p_2, p_3, p_4.
Draw their locations in the complex plane.

$$p_{1,2} = \pm\sqrt{-\lambda_1} = \pm\sqrt{-9} = \pm 3\,i$$

$$p_{3,4} = \pm\sqrt{-\lambda_2} = \pm\sqrt{-11} = \pm 3.32\,i$$

The solution is **stable**/$\boxed{\textbf{neutrally stable}}$/**unstable**.





(e) Write the solution for $X(t)$ in terms of the sine and cosine functions, and t.
Result: (also in terms of the yet-to-be-determined constants c_1, c_2, c_3, c_4)

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \{c_1 \sin(3\,t) + c_2 \cos(3\,t)\} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \{c_3 \sin(3.32\,t) + c_4 \cos(3.32\,t)\}$$

(f) Determine c_1, c_2, c_3, c_4 when $x(0) = 0.2$, $\theta(0) = 0$, $\dot{x}(0) = 0$, $\dot{\theta}(0) = 0$.
Write explicit solutions for $x(t)$ and $\theta(t)$ using the initial values
Result: $c_1 = 0 \qquad c_2 = -0.1 \qquad c_3 = 0 \qquad c_4 = 0.1$

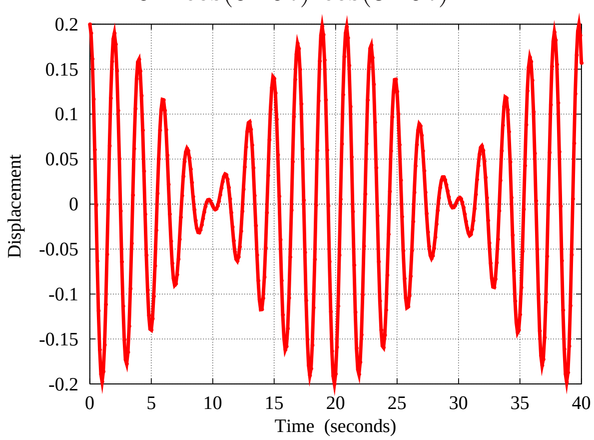
$$x(t) = 0.1 \cos(3\,t) + 0.1 \cos(3.32\,t)$$

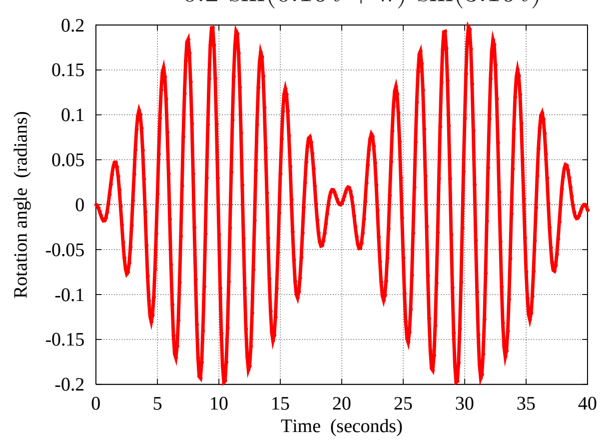
$$\theta(t) = -0.1 \cos(3\,t) + 0.1 \cos(3.32\,t)$$

(g) Using trigonometric identities, equation (1.20), and $\cos(a) = -\cos(a{+}\pi)$, it can be shown that the previous solution for $x(t)$ and $\theta(t)$ is (you do not need to show this)

$$x(t) = 0.2 \sin(-0.16\,t + \frac{\pi}{2}) \sin(3.16\,t + \frac{\pi}{2}) = 0.2 \cos(0.16\,t) \cos(3.16\,t)$$

$$\theta(t) = 0.2 \sin(-0.16\,t) \sin(3.16\,t) = 0.2 \sin(0.16\,t + \pi) \sin(3.16\,t)$$





Interpret the time-behavior of $x(t)$ and $\theta(t)$.
Both $x(t)$ and $\theta(t)$ exhibit the ***beat phenomena*** with a high-frequency of $3.16\ \frac{\text{rad}}{\text{sec}}$ and a low-frequency of $0.16\ \frac{\text{rad}}{\text{sec}}$.
Since $x(t)$ and $\theta(t)$ are ***coupled***, x's maximum amplitude coincides with θ's minimum amplitude (and vice-versa).