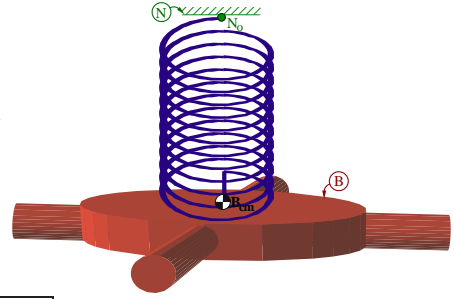


13.6 Coupled motions of WilberForce pendulum

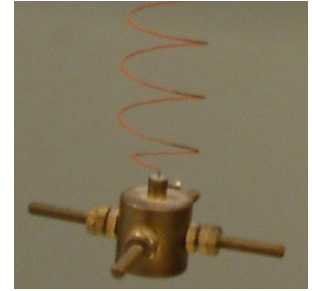
A rigid body B is attached to a spring at B_{cm} (B 's center of mass). The spring's other end is attached to point N_o fixed in a Newtonian reference frame N . This system's motion is governed by

$$\begin{aligned} m \ddot{x} + k_x x + k_c \theta &= 0 \\ I \ddot{\theta} + k_c x + k_\theta \theta &= 0 \end{aligned}$$



Quantity	Symbol	Type
B 's mass	m	Positive constant
B 's central moment of inertia about vertical axis	I	Positive constant
Extensional linear spring constant modeling	k_x	Positive constant
Torsional linear spring constant modeling	k_θ	Positive constant
Coupled linear spring constant modeling	k_c	Positive constant
Translational stretch of spring from equilibrium	x	Variable
Rotational stretch of spring from equilibrium	θ	Variable

Purchase Wilburforce pendulum at PASCO Scientific or B&B Co. (Super Spinnerama). More WilberForce information at "WilberForce pendulum oscillations and normal modes", Berg and Marshall, Am. J. Physics Vol. 59, No. 1, January 1991.



- (a) Write the ODEs in the matrix form $M \ddot{X} + B \dot{X} + K X = [0]$. Next, substitute the assumed solution $X(t) = U e^{pt}$ where p is a constant (to-be-determined) and U is a **non-zero** 2×1 matrix of constants (to-be-determined). Show p and U are governed by a **generalized eigenvalue problem**. Next, define $\lambda \triangleq -p^2$ and find a **scalar** equation that governs λ .

Result:

$$\underbrace{\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}}_M \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}}_B \underbrace{\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}}_{\dot{X}} + \underbrace{\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}}_K \underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow [p^2 M + K] U e^{pt} = [0] \Rightarrow [p^2 M + K] U = [0]$$

$$\lambda \triangleq -p^2 \Rightarrow \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} U = [0] \Rightarrow \det \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} = 0$$

- (b) Show this can be recast as a **standard eigenvalue problem** $[-\lambda I + A]U = [0]$.

Result – determine A: (note I is the 2×2 identity matrix)

$$[-\lambda I + A]U = [0] \quad \text{where} \quad A = M^{-1}K = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

- (c) Complete the two blanks in the following polynomial equation that governs $\lambda \triangleq -p^2$.

Result: (Note: The blanks only involve m, I, k_x, k_θ, k_c).

$$\lambda^2 + \square * \lambda + \square = 0$$

- (d) For certain values of m, I, k_x, k_θ, k_c , the matrix $A \triangleq M^{-1}K = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$.

Find the **eigenvalues** and corresponding **eigenvectors** of A .

Result:

$$\lambda_1 = 9 \qquad \lambda_2 = \boxed{}$$

$$U_1 = \begin{bmatrix} \boxed{} \\ 1 \end{bmatrix} \qquad U_2 = \begin{bmatrix} \boxed{} \\ 1 \end{bmatrix}$$

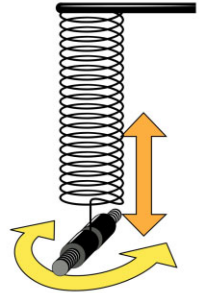
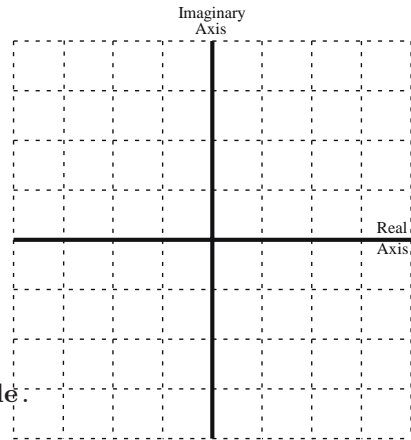
Calculate the values of p_1, p_2, p_3, p_4 .

Draw their locations in the complex plane.

$$p_{1,2} = \boxed{}$$

$$p_{3,4} = \boxed{}$$

The solution is **stable/neutrally stable/unstable**:



- (e) Write the solution for $X(t)$ in terms of the yet-to-be-determined constants c_1, c_2, c_3, c_4 , the sine and cosine functions, and t . Next, write explicit solutions for $x(t)$ and $\theta(t)$ using the initial values $x(0) = 0.2, \theta(0) = 0, \dot{x}(0) = 0, \dot{\theta}(0) = 0$.

Result:

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} \boxed{} \\ 1 \end{bmatrix} \{c_1 \sin(\boxed{} t) + c_2 \cos(\boxed{} t)\} + \begin{bmatrix} \boxed{} \\ 1 \end{bmatrix} \{c_3 \sin(\boxed{} t) + c_4 \cos(\boxed{} t)\}$$

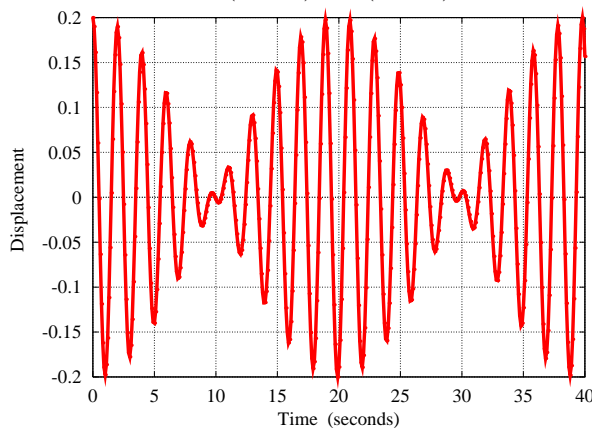
$$x(t) = \boxed{} \cos(3t) + \boxed{} \cos(\boxed{} t)$$

$$\theta(t) = \boxed{} \cos(3t) + \boxed{} \cos(\boxed{} t)$$

- (f) Using trigonometric identities, equation (1.??), and $\cos(a) = -\cos(a+\pi)$, it can be shown that the previous solution for $x(t)$ and $\theta(t)$ is (you do not need to show this)

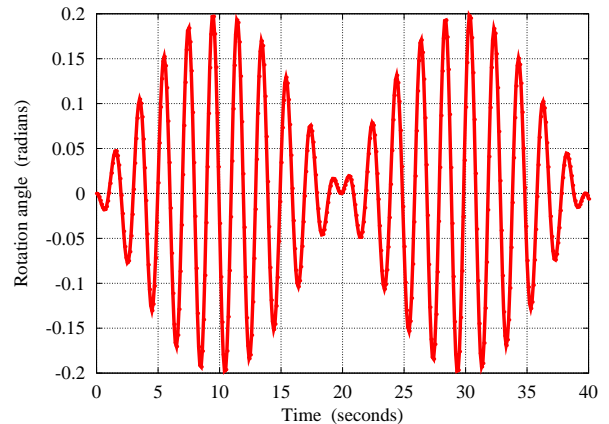
$$x(t) = 0.2 \sin(-0.16t + \frac{\pi}{2}) \sin(3.16t + \frac{\pi}{2})$$

$$= 0.2 \cos(0.16t) \cos(3.16t)$$



$$\theta(t) = 0.2 \sin(-0.16t) \sin(3.16t)$$

$$= 0.2 \sin(0.16t + \pi) \sin(3.16t)$$



Interpret the time-behavior of $x(t)$ and $\theta(t)$.

Both $x(t)$ and $\theta(t)$ exhibit the **beat phenomena** with a high-frequency of $\boxed{} \frac{\text{rad}}{\text{sec}}$ and a low-frequency of $\boxed{} \frac{\text{rad}}{\text{sec}}$. Since $x(t)$ and $\theta(t)$ are **coupled**, x 's maximum amplitude coincides with θ 's minimum amplitude (and vice-versa).