Homework 1. Chapter 2.

Basis independent vector operations: $-\vec{b}$ $\quad s\,\vec{b}$ $\quad \vec{a}+\vec{b}$ $\quad \angle(\vec{a},\vec{b})$ $\quad \vec{a}\cdot\vec{b}$ $\quad \vec{a}\times\vec{b}$

Show work – except for ♣ fill-in-blanks (print .pdf from www.MotionGenesis.com ⇒ Textbooks ⇒ Resources).

1.1 ♣ Solving problems – what physicists and engineers do.

Understanding dynamics results from **doing** problems. Many problems herein guide you to help you synthesize processes (imitation). Please **do** these problems by yourself or with colleagues/instructors and use the textbook and other resources.

Confucius 500 B.C.
"I hear and I forget.
I see and I remember.
I _____ and I understand."

"By three methods we may learn wisdom:
1^{st} by reflection, which is noblest;
2^{nd} by imitation, which is easiest;
3^{rd} by experience, which is the bitterest."

1.2 ♣ What is a vector (as defined by Gibbs circa 1897)? (Section 2.2)

Two properties (attributes) of a vector are ________ and ________ (fill in the blanks).

1.3 ♣ What is a zero vector? (Section 2.3)

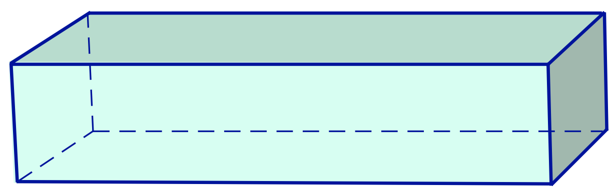
A zero vector $\vec{0}$ has a magnitude of 0 ($|\vec{0}| = 0$). **True/False** (circle true or false).
A zero vector $\vec{0}$ has a direction. **True/False**
any$\vec{\text{V}}$ector $+\ \vec{0}\ =$ **any$\vec{\text{V}}$ector** **True/False**

1.4 ♣ Unit vectors. (Section 2.4)

All unit vectors have a magnitude of 1 (e.g., $|\hat{\mathbf{i}}| = 1$, $|\hat{\mathbf{j}}| = 1$, $|\hat{\mathbf{k}}| = 1$).	**True/False**
Typically, a unit vector is denoted with a hat, e.g., as $\hat{\mathbf{k}}$ rather than $\vec{\mathbf{k}}$.	**True/False**
All unit vectors are equal.	**True/False**
A unit vector $\hat{\mathbf{u}}$ in the direction of the non-zero vector $\vec{\mathbf{v}}$ is $\hat{\mathbf{u}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$.	**True/False**

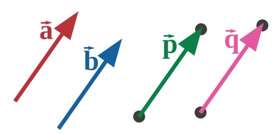
1.5 ♣ Draw the vectors $\vec{a}$, $\vec{b}$, $\hat{c}$, $\hat{d}$ (Section 2.2)

$\vec{a}$	Horizontally-right vector.
$\vec{b}$	Vertically-upward vector.
$\hat{c}$	Outwardly-directed **unit** vector.
$\hat{d}$	Inwardly-directed **unit** vector.



1.6 ♣ Equal vectors? Equal position vectors? (Section 2.5)

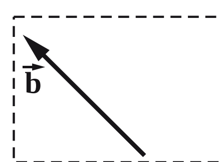
For the generic vectors $\vec{a}$ and $\vec{b}$ shown right, $\vec{a} = \vec{b}$ **True/False**.
For the position vectors $\vec{p}$ and $\vec{q}$ shown right, $\vec{p} = \vec{q}$ **True/False**.



1.7 ♣ Negating a vector. (Section 2.8)

Draw the vector $-\vec{b}$. Negating the vector $\vec{b}$ results in a vector with different:

magnitude direction orientation sense (circle **all** that apply)

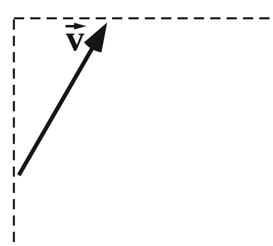


Historical note: Negative numbers (e.g., -3) were not widely accepted until 1800 A.D.

1.8 ♣ Vector magnitude and direction (orientation and sense). (Section 2.2)

The figure to the right shows a vector $\vec{v}$. **Draw** the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$.

$\vec{a}$	Same magnitude and direction as $\vec{v}$ ($\vec{a} = \vec{v}$).
$\vec{b}$	Same magnitude as $\vec{v}$, with $\vec{b} = -\vec{v}$ (**antiparallel**).
$\vec{c}$	Same magnitude as $\vec{v}$, but different direction with $\vec{c} \neq -\vec{v}$.
$\vec{d}$	Smaller magnitude than $\vec{v}$, but same direction as $\vec{v}$.
$\vec{e}$	Different magnitude and different direction than $\vec{v}$.



1.9 ♣ **Vector magnitude and direction.** **(Section 2.2)**

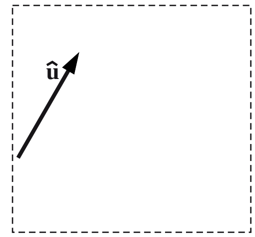
Knowing x is a real number (e.g., -3 or 0 or 7.8) and $\hat{\mathbf{u}}$ is a horizontal unit vector ⟶, complete ***magnitude*** with $\le$ or $\ge$ and complete ***direction*** with $+\hat{\mathbf{u}}$ or $-\hat{\mathbf{u}}$.

Vector	with	Magnitude	Direction
$x\,\hat{\mathbf{u}}$	$x \ge 0$	$\lvert x\,\hat{\mathbf{u}} \rvert$ $\ge$ 0	$+\hat{\mathbf{u}}$
$x\,\hat{\mathbf{u}}$	$x \le 0$	$\lvert x\,\hat{\mathbf{u}} \rvert$ ____ 0	____
$-x\,\hat{\mathbf{u}}$	$x \ge 0$	$\lvert -x\,\hat{\mathbf{u}} \rvert$ ____ 0	____
$-x\,\hat{\mathbf{u}}$	$x \le 0$	$\lvert -x\,\hat{\mathbf{u}} \rvert$ ____ 0	____

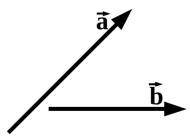
1.10 ♣ **Multiplying a vector by a scalar.** **(Section 2.7)**

The following statements involve a unit vector $\hat{\mathbf{u}}$ and a real scalar s ($s \neq 0$).
If a statement is **true**, provide any numerical value for s that supports your answer, and if **true** also **draw** a corresponding vector, i.e., $\vec{\mathbf{a}}$ or $\vec{\mathbf{b}}$ or $\vec{\mathbf{c}}$.

$s\,\hat{\mathbf{u}}$ can have a different ***magnitude*** than $\hat{\mathbf{u}}$.	If **true** $s =$ ____, **draw** $\vec{\mathbf{a}}$.
$s\,\hat{\mathbf{u}}$ can have a different ***direction*** than $\hat{\mathbf{u}}$.	If **true** $s =$ ____, **draw** $\vec{\mathbf{b}}$.
$s\,\hat{\mathbf{u}}$ can have different ***magnitude and direction*** than $\hat{\mathbf{u}}$.	If **true** $s =$ ____, **draw** $\vec{\mathbf{c}}$.



1.11 ♣ **Graphical vector addition/subtraction.** **(Sections 2.6, 2.8)**

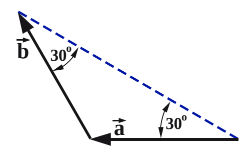


Draw $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ | **Draw** $\vec{\mathbf{b}} + \vec{\mathbf{a}}$ | **Draw** $\vec{\mathbf{a}} + -\vec{\mathbf{b}}$ | **Draw** $\vec{\mathbf{b}} - \vec{\mathbf{a}}$ | **Draw** $-\vec{\mathbf{a}} - \vec{\mathbf{b}}$

1.12 ♣ **Angle $\angle(\vec{\mathbf{a}}, \vec{\mathbf{b}})$ between vectors.** **(Section 2.9)**

For the figure shown right, determine the numerical value for the angle between vector $\vec{\mathbf{a}}$ and vector $\vec{\mathbf{b}}$.

Result: $\angle(\vec{\mathbf{a}},\ \vec{\mathbf{b}}) =$ ____°



1.13 ♣ **Visual representation of a vector dot-product.** **(Section 2.9)**

Write the ***definition*** of the dot-product of a vector $\vec{\mathbf{a}}$ with a vector $\vec{\mathbf{b}}$.
Include a ***sketch*** with ***each symbol*** in your definition clearly labeled.

Result: $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \triangleq$ ____ ____ ____

Sketch should include $\vec{\mathbf{a}},\ \vec{\mathbf{b}},\ |\vec{\mathbf{a}}|,\ |\vec{\mathbf{b}}|,\ \theta$.

Knowing $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are arbitrary vectors, complete the blanks with $\le$, $=$, or $\ge$.

When the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $0°$	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ ____ 0	(**parallel**)
When the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $90°$	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ ____ 0	(**perpendicular**)
When the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $180°$	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ ____ 0	(**antiparallel**)
For arbitrary vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$,	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ ____ $\vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$	

1.14 ♣ **Visual representation of a vector cross-product.** **(Section 2.10)**

Write the ***definition*** of the cross-product of a vector $\vec{\mathbf{a}}$ with a vector $\vec{\mathbf{b}}$.
Include a ***sketch*** with ***each symbol*** in your definition clearly labeled.

Result: $\vec{\mathbf{a}} \times \vec{\mathbf{b}} \triangleq$ ____ ____ ____$(\theta)\ \hat{\mathbf{u}}$

where $\hat{\mathbf{u}}$ is ____________________

____________________ and θ is ____________________

Sketch should include $\vec{\mathbf{a}},\ \vec{\mathbf{b}},\ |\vec{\mathbf{a}}|,\ |\vec{\mathbf{b}}|,\ \theta,\ \hat{\mathbf{u}}$.

Knowing $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are non-zero vectors, complete the blanks with $=$ or $\neq$.

When the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $0°$	$\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ ____ $\vec{\mathbf{0}}$	(**parallel**)
When the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $90°$	$\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ ____ $\vec{\mathbf{0}}$	(**perpendicular**)
When the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $180°$	$\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ ____ $\vec{\mathbf{0}}$	(**antiparallel**)
For arbitrary vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$,	$\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ ____ $\vec{\mathbf{b}} \times \vec{\mathbf{a}}$	

1.15 Properties of vector dot/cross-products Draw/show work. $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$. (Sections 2.9.1, 2.10)

When $\vec{a}$ is *parallel* to $\vec{b}$,	$\vec{a} \cdot \vec{b} = 0$	True/False	$\vec{a} \times \vec{b} = \vec{0}$	True/False
When $\vec{a}$ is *perpendicular* to $\vec{b}$,	$\vec{a} \cdot \vec{b} = 0$	True/False	$\vec{a} \times \vec{b} = \vec{0}$	True/False
For arbitrary vectors $\vec{a}$ and $\vec{b}$,	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	True/False	$\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$	True/False

1.16 Dot-products and cross-products via definitions. Show work. (Sections 2.9, 2.10)

- **Draw** a unit vector $\widehat{\mathbf{k}}$ outward-normal to the plane of the paper (perpendicular to $\vec{a}$ and $\vec{b}$).
- **Redraw** each figure to clarify $\angle(\vec{a}, \vec{b})$, the angle between $\vec{a}$ and $\vec{b}$ (useful for dot and cross-product).
- Knowing $|\vec{a}| = 2$ and $|\vec{b}| = 4$, calculate each expressions below (2^+ significant digits) using only the definitions of dot-product and cross-product.

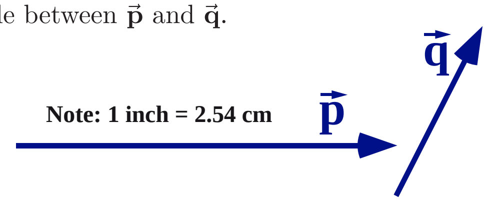
$\angle(\vec{a}, \vec{b}) = \_\_\_^\circ$	$\angle(\vec{a}, \vec{b}) = \_\_\_^\circ$	$\angle(\vec{a}, \vec{b}) = \_\_\_^\circ$	$\angle(\vec{a}, \vec{b}) = \_\_\_^\circ$
$\vec{a} \cdot \vec{b} = \_\_\_$	$\vec{a} \cdot \vec{b} = \_\_\_$	$\vec{a} \cdot \vec{b} = \_\_\_$	$\vec{a} \cdot \vec{b} = \_\_\_$
$\vec{a} \times \vec{b} = -8\,\widehat{\mathbf{k}}$	$\vec{a} \times \vec{b} = \_\_\_$	$\vec{a} \times \vec{b} = \_\_\_$	$\vec{a} \times \vec{b} = \_\_\_$

1.17 Visual estimation of vector dot/cross-products. Show work. (Sections 2.9, 2.10)

Estimate the magnitude of the vector $\vec{q}$ shown below, the angle between $\vec{p}$ and $\vec{q}$, $\vec{p} \cdot \vec{q}$, and the magnitude of $\vec{p} \times \vec{q}$. **Show work** and **redraw** to clarify the angle between $\vec{p}$ and $\vec{q}$.

Result: (Provide numerical results with 1 or more significant digits).

$\lvert\vec{p}\rvert \approx$ 4.0 cm	$\lvert\vec{q}\rvert \approx$ ___ cm	$\angle(\vec{p}, \vec{q}) \approx$ ___ °
$\vec{p} \cdot \vec{q} \approx$ ___ cm^2		$\lvert\vec{p} \times \vec{q}\rvert \approx$ ___ cm^2



1.18 ♣ Vector operations and units. (Chapter 2)

Each vector operation below involves a position vector $\vec{r}$ (with **units** of m) and/or a velocity vector $\vec{v}$ (with **units** of $\frac{m}{s}$). Determine whether the operation produces a well-defined scalar or vector or is **undefined**. If well-defined, determine the associated units.

Operation:	$-\vec{r}$	$5\,\vec{v}$	$5\,\frac{m}{s} + \vec{v}$	$\vec{r} + 2\,\vec{r}$	$\vec{r} + \vec{v}$	$5\,\frac{m}{s} \cdot \vec{v}$	$\vec{r} \cdot \vec{v}$	$\vec{r} \times \vec{v}$
Produces:	vector							
Units:	meters							

1.19 ♣ Vector exponentiation: $\vec{v}^{\,2} = \vec{v} \cdot \vec{v}$ and $\vec{v}^{\,3}$. (Section 2.9)

The following is a reasonable proof that $\vec{v}^{\,2} = \vec{v} \cdot \vec{v}$. **True/False** (if **False**, provide a proof).

$$\vec{v}^{\,2} \triangleq |\vec{v}|^2 \qquad \vec{v} \cdot \vec{v} \underset{(2.2)}{\triangleq} |\vec{v}|\,|\vec{v}|\,\cos(0^\circ) = |\vec{v}|^2 \qquad \vec{v}^{\,2} = \vec{v} \cdot \vec{v}$$

Complete the proof that relates $\vec{v}^{\,3}$ to $\vec{v} \cdot \vec{v}$ raised to a real number.

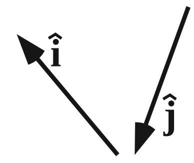
Result:
$$|\vec{v}| \underset{(2.4)}{=} \sqrt{\square \cdot \square} \qquad \vec{v}^{\,3} \triangleq |\vec{v}|^{\square} = \left(\sqrt{\square \cdot \square}\right)^{\square} = (\vec{v} \cdot \vec{v})^{\frac{3}{2}}$$

1.20 ♣ $|c\,\widehat{\mathbf{a}}_x|$ Calculate vector magnitude with dot products. (Section 2.9 and Hw 1.19)

Show how the vector dot-product can be used to show that the magnitude of the vector $c\,\widehat{\mathbf{a}}_x$ (c is a positive or **negative** number and $\widehat{\mathbf{a}}_x$ is a unit vector) can be written solely in terms of c (without $\widehat{\mathbf{a}}_x$).

Result:
$$|c\,\widehat{\mathbf{a}}_x| = {}^{+}\sqrt{\square \cdot \square} = {}^{+}\sqrt{c^2 * \square \cdot \square} = {}^{+}\sqrt{c^2} = \text{abs}(c)$$

1.21 †(Challenge) **Magnitude of the vector $\vec{\mathbf{v}}$.** **Show work.** **(Section 2.9)**



Knowing the angle between a unit vector $\hat{\mathbf{i}}$ and unit vector $\hat{\mathbf{j}}$ is 120°, calculate a numerical value for the magnitude of $\vec{\mathbf{v}} = 3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$.

Result: $|\vec{\mathbf{v}}| = \sqrt{\boxed{13}}$ Note: The answer is ***not*** $\sqrt{25} = 5$.

1.22 ♣ **Property of scalar triple product.** **(Section 2.11)**

For arbitrary non-zero vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$: $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}$ **Never/Sometimes/Always**

A property of the ***scalar triple product*** is $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{a}} = 0$. **True/False**.

1.23 ♣ **Property of vector triple cross-product.** **(Sections 2.10, 2.11)**

Complete the following equation: $\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\,(\;\_\_\_\_\;) - \vec{\mathbf{c}}\,(\;\_\_\_\_\;)$

For arbitrary vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$: $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times (\vec{\mathbf{a}} \times \vec{\mathbf{c}})$ **True/False** (show work).

1.24 ♣ **Form the *unit* vector $\hat{\mathbf{u}}$ having the same direction as $c\,\hat{\mathbf{a}}_x$.** **(Section 2.4)**

Result: $\hat{\mathbf{u}} = \dfrac{\_\_}{\_\_\_\_}\,\hat{\mathbf{a}}_x$ Note: $\hat{\mathbf{a}}_x$ is a unit vector and c is a non-zero real number, e.g., 3 or -3.

1.25 ♣ **Coefficient of $\hat{\mathbf{u}}$ in cross products – definitions and trig functions.** **(Section 2.10)**

The ***cross product*** of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ can be written in terms of a real scalar s as $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = s\,\hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ is a unit vector perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in a direction defined by the **right-hand rule**. The coefficient s of the unit vector $\hat{\mathbf{u}}$ is inherently non-negative. **True/False**.

1.26 ♣ **Ranges of angles from dot-product and cross-product calculations.** **(Sections 2.9, 2.10)**

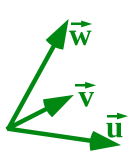
Quantity	Numerical range of values
$c = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ (assume $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are known so a numerical value for $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ can be calculated).	__ $\le c \le$ __
$s = \lvert\hat{\mathbf{a}} \times \hat{\mathbf{b}}\rvert$ (assume $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are known so a numerical value for $\lvert\hat{\mathbf{a}} \times \hat{\mathbf{b}}\rvert$ can be calculated).	__ $\le s \le$ __
Angle θ_c between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ that can be uniquely determined **solely** from c. Use the principal range available from a simple calculator's inverse sine and inverse cosine.	__° $\le \theta_c \le$ __°
Angle θ_s between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ that can be uniquely determined **solely** from s. Use the principal range available from a simple calculator's inverse sine and inverse cosine.	__° $\le \theta_s \le$ __°
Angle θ between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, i.e., $\theta = \angle(\hat{\mathbf{a}}, \hat{\mathbf{b}})$	__° $\le \theta \le$ __°

Note: The range of θ_s is smaller than the range for θ. Hence, s and θ_s are insufficient to correctly calculate θ.
What this means: Use the **dot-product ·** to calculate an angle θ from two given/known vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

1.27 ♣ **Using vector identities to simplify expressions** **(refer to Homework 1.15)**

One reason to treat vectors as ***basis-independent*** quantities is to simplify vector expressions ***without*** resolving the vectors into orthogonal "$\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}}$" or "$\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}$" components. Simplify the following vector expressions using mathematical properties of dot-products and cross-products.

Express results in terms of dot-products · and cross-products × of the arbitrary vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$.
$\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ are not necessarily orthogonal or coplanar.



Vector expression	Simplified vector expression
$(3\,\vec{\mathbf{u}} - 2\,\vec{\mathbf{v}}) \times (\vec{\mathbf{u}} + \vec{\mathbf{v}})$	__ $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$
$(3\,\vec{\mathbf{u}} - 2\,\vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} + \vec{\mathbf{v}})$	__ $\vec{\mathbf{u}}^2$ − __ $\vec{\mathbf{v}}^2$ + __ $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$
$(\vec{\mathbf{u}} - \vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} + \vec{\mathbf{v}})$	____ − ____
$(3\,\vec{\mathbf{u}} - 2\,\vec{\mathbf{v}}) \times (\vec{\mathbf{u}} + \vec{\mathbf{v}}) \cdot (2\,\vec{\mathbf{u}} - 7\,\vec{\mathbf{v}})$	__
$(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \times (\vec{\mathbf{v}} + 2\,\vec{\mathbf{w}}) \cdot (\vec{\mathbf{w}} + 2\,\vec{\mathbf{u}})$	___ $\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$

1.28 ♣ Vector concepts: Solving a vector equation? (Section 2.9.3)

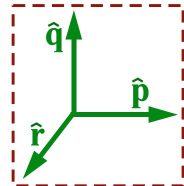
Shown right is a vector equation and a questionable process that solves for v_x ($\widehat{\mathbf{a}}_x$ is a unit vector and v_x, $\dot{\theta}$, R are scalars).

$$v_x \, \widehat{\mathbf{a}}_x \;=\; \dot{\theta} \, R \, \widehat{\mathbf{a}}_x$$
$$v_x \;=\; \dot{\theta} \, R \, \frac{\widehat{\mathbf{a}}_x}{\widehat{\mathbf{a}}_x} \;=\; \dot{\theta} \, R$$

This is a valid process to solve for v_x. **True/False**.

Explain: ______

1.29 Change a vector equation to scalar equations. Show work. (Section 2.9.3)



Shown right are three mutually orthogonal unit vectors $\widehat{\mathbf{p}}$, $\widehat{\mathbf{q}}$, $\widehat{\mathbf{r}}$.
Use a vector operation (e.g., +, ∗, ·, ×) to change the **vector** equation $(2x-4)\,\widehat{\mathbf{p}} \;=\; \vec{\mathbf{0}}$ into **one scalar** equation and subsequently solve the scalar equation for x.

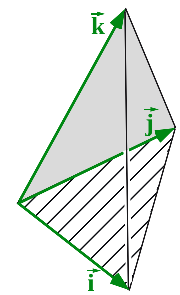
Result: $(2x-4)\,\widehat{\mathbf{p}} \;=\; \vec{\mathbf{0}} \;\overset{??}{\Rightarrow}\; (2x-4) \;=\; 0 \;\Rightarrow\; x \;=\; 2$

Show ***every*** vector operation (e.g., +, ∗, ·, ×) that changes the following **vector** equation into **three scalar** equations and subsequently solve the scalar equations for x, y, z.

$$(2\,x-4)\,\widehat{\mathbf{p}} \;+\; (3\,y-9)\,\widehat{\mathbf{q}} \;+\; (4\,z-16)\,\widehat{\mathbf{r}} \;=\; \vec{\mathbf{0}}$$

Result: $(2\,x-4) \;=\; 0$ $\quad (3\,y-9) \;=\; 0$ $\quad (\;\_\_\_\_\;) \;=\; 0$
$x \;=\; 2$ $\quad y \;=\; 3$ $\quad z \;=\; 4$

†**Optional:** The figure to the right shows three **non-orthogonal**, non-coplanar vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$. Show **every** vector operation that changes the following **vector** equation into **three** uncoupled **scalar** equations and subsequently solve those scalar equations for x, y, z.



$$(2\,x-4)\,\vec{\mathbf{i}} \;+\; (3\,y-9)\,\vec{\mathbf{j}} \;+\; (4\,z-16)\,\vec{\mathbf{k}} \;=\; \vec{\mathbf{0}}$$

Result: $(2\,x-4) \;=\; 0$ $\quad (3\,y-9) \;=\; 0$ $\quad (\;\_\_\_\_\;) \;=\; 0$
$x \;=\; 2$ $\quad y \;=\; 3$ $\quad z \;=\; 4$

Hint: think × ·, not matrix algebra.

1.30 ♣ Number of independent scalar equations from 1 vector equation. (Section 2.9.3)

The **vector** equation shown right is useful for static analyses of a system S. $\quad \vec{\mathbf{F}}^{\,S} \;=\; \vec{\mathbf{0}}$

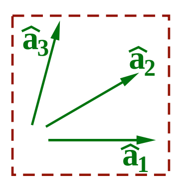
In the table to the right, box **all** integers that could be equal to the number of ***independent scalar*** equations produced by the previous vector equation. Hint: Hw 1.29. Related Hw 13.15.

System type	Integer(s)
1D (line)	0 1 2 3 4^+
2D (planar)	0 1 2 3 4^+
3D (spatial)	0 1 2 3 4^+

Note: 1D/linear means $\vec{\mathbf{F}}^{\,S}$ can be expressed in terms of one vector $\widehat{\mathbf{i}}$.
2D/planar means $\vec{\mathbf{F}}^{\,S}$ can be expressed in terms of two non-parallel unit vectors $\widehat{\mathbf{i}}$ and $\widehat{\mathbf{j}}$.
3D/spatial means $\vec{\mathbf{F}}^{\,S}$ can be expressed in terms of three non-coplanar unit vectors $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, $\widehat{\mathbf{k}}$.

1.31 ♣ Vector concepts: Solving a vector equation (just circle true or false and fill-in the blank).

Consider the following vector equation written in terms of the scalars x, y, z and three unique non-orthogonal **coplanar** unit vectors $\widehat{\mathbf{a}}_1$, $\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_3$.



$$(2\,x-4)\,\widehat{\mathbf{a}}_1 \;+\; (3\,y-9)\,\widehat{\mathbf{a}}_2 \;+\; (4\,z-16)\,\widehat{\mathbf{a}}_3 \;=\; \vec{\mathbf{0}}$$

The **unique** solution to this vector equation is $x=2$, $y=3$, $z=4$. **True/False**.

Explain: $\widehat{\mathbf{a}}_2$ can be expressed in terms of $\widehat{\mathbf{a}}_1$ and $\widehat{\mathbf{a}}_3$ (i.e., $\widehat{\mathbf{a}}_2$ is a linear combination of $\widehat{\mathbf{a}}_1$ and $\widehat{\mathbf{a}}_3$).
Hence the vector equation produces ______ linearly independent scalar equations.

1.32 ♣ Gibbs vectors (≈1900 AD) revolutionizes Euclidean geometry (300 BC).

For each geometrical quantity shown right, circle the vector operation(s) (dot-product, cross-product, or both) that is ***most*** useful for their calculation.

Distance: • × Section 2.9.2	Angle: • × Section 2.9.2
Area: • × Section 2.10.1	Volume: • × Section 2.11.1

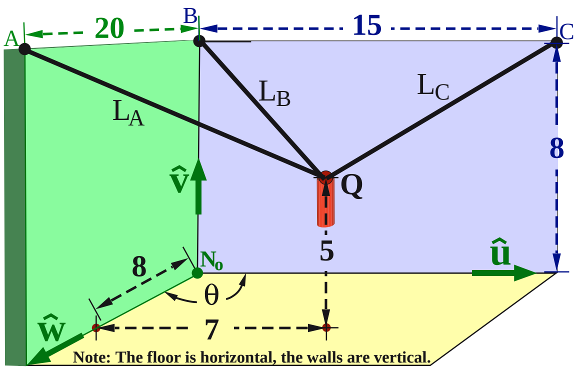
1.33 ♣ Order of operations with vector dot products (•) and cross products (×). (Chapter 2)

Create a valid expression by adding parentheses to each expression or ~~cross-out~~ the expression if it is inherently invalid.
Example: $3 * \vec{a} + \vec{b} \;\Rightarrow\; (3 * \vec{a}) + \vec{b}$.

$\vec{a} \cdot \vec{b} + \vec{c}$	$\vec{a} \cdot \vec{b} \times \vec{c}$	$\vec{a} + 5 \times \vec{c}$
$\vec{a} \times \vec{b} + \vec{c}$	$\vec{a} \times \vec{b} \cdot \vec{c}$	$\vec{a} \cdot \vec{b} \cdot \vec{c}$

1.34 †Microphone cable lengths (non-orthogonal walls) "It's just geometry". Show work.

- A microphone Q is attached to three pegs A, B, C by three cables. Knowing the peg locations, microphone location, and the angle θ between the vertical walls, express L_A, L_B, L_C solely in terms of numbers and θ. Next, complete the table by calculating L_B when $\theta = 120°$.



Distance between A and B	20 m
Distance between B and C	15 m
Distance between N_o and B	8 m
Distance along back wall (see picture)	7 m
Q's height above N_o	5 m
Distance along side wall (see picture)	8 m
L_A: Length of cable joining A and Q	16.9 m
L_B: Length of cable joining B and Q	8.1 m
L_C: Length of cable joining C and Q	14.2 m

$${}^{N_o}\vec{r}^{\,Q} = 7\,\hat{u} + 5\,\hat{v} + 8\,\hat{w}$$

Result: $L_A = \sqrt{202 - \underline{\quad}\;\cos(\theta)}$ $\quad L_B = \sqrt{122 + 112\,\cos(\theta)}$ $\quad L_C = \sqrt{\underline{\quad} - 128\,\underline{\quad}}$

Hint: To do this efficiently, use only unit vectors $\hat{u}$, $\hat{v}$, $\hat{w}$.
Hint: Use the distributive property of the vector dot-product as shown in Section 2.9.1 and Hw 2.4.
Note: Synthesis problems are difficult. Think, talk, draw, sleep, walk, get help, . . . (if needed, read Section 3.3).

Vocabulary: This is ***inverse kinematics***. The position of "end-effector" Q is known and you determine the cable lengths.

- Using a dot-product, show the angle β between lines $\overline{B\,N_o}$ and $\overline{B\,Q}$ is $\beta \approx$ 68.33°.
  Optional: Verify the calculation of β using the law of cosines.

Vector addition, dot products, and cross products: $+ \quad \cdot \quad \times$

Show work – except for ♣ fill-in-blanks (print .pdf from www.MotionGenesis.com ⇒ Textbooks ⇒ Resources).

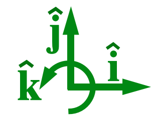
2.1 ♣ Right-handed, orthogonal, unit vectors. (Section 4.1)

Draw a set of right-handed orthogonal (mutually perpendicular) unit vectors consisting of $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$. In other words, draw $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ so that $\widehat{\mathbf{n}}_y$ is perpendicular (orthogonal) to $\widehat{\mathbf{n}}_x$ and $\widehat{\mathbf{n}}_z = \widehat{\mathbf{n}}_x \times \widehat{\mathbf{n}}_y$.

2.2 ♣ Adding and subtracting vectors. (Sections 2.6, 2.8)

Given: Vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ expressed in terms of unit vectors $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, $\widehat{\mathbf{k}}$. Form the vector sums and differences below.

$$\vec{\mathbf{p}} = a\,\widehat{\mathbf{i}} + b\,\widehat{\mathbf{j}} + c\,\widehat{\mathbf{k}}$$
$$\vec{\mathbf{q}} = x\,\widehat{\mathbf{i}} + y\,\widehat{\mathbf{j}} + z\,\widehat{\mathbf{k}}$$



Result: $\vec{\mathbf{p}} + \vec{\mathbf{q}} = (a+x)\,\widehat{\mathbf{i}} + (\boxed{\quad})\,\widehat{\mathbf{j}} + (\boxed{\quad})\,\widehat{\mathbf{k}}$ $\qquad$ $\vec{\mathbf{p}} - \vec{\mathbf{q}} = (a-x)\,\widehat{\mathbf{i}} + (\boxed{\quad})\,\widehat{\mathbf{j}} + (\boxed{\quad})\,\widehat{\mathbf{k}}$

2.3 ♣ Words: Physical vectors and column matrices. (Section 2.1, Hw 1.2)

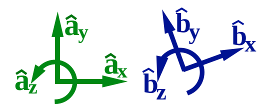
True/False As defined by Gibbs and for $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$, physical vectors have magnitude and direction.

True/False In math (linear algebra), a column matrix is called a "vector".

True/False The physical vector $\widehat{\mathbf{a}}_x + 2\,\widehat{\mathbf{a}}_y + 3\,\widehat{\mathbf{a}}_z$ can be written $[\widehat{\mathbf{a}}_x \;\; \widehat{\mathbf{a}}_y \;\; \widehat{\mathbf{a}}_z] * \begin{bmatrix}1\\2\\3\end{bmatrix}$.
Note: $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ are the orthogonal unit vectors shown below.

True/False The physical vector $\widehat{\mathbf{a}}_x + 2\,\widehat{\mathbf{a}}_y + 3\,\widehat{\mathbf{a}}_z$ is equal to the column matrix $\begin{bmatrix}1\\2\\3\end{bmatrix}$.

True/False $\widehat{\mathbf{a}}_x + 2\,\widehat{\mathbf{a}}_y + 3\,\widehat{\mathbf{a}}_z \;+\; 4\,\widehat{\mathbf{b}}_x + 5\,\widehat{\mathbf{b}}_y + 6\,\widehat{\mathbf{b}}_z = \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}5\\7\\9\end{bmatrix}$
($\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are shown right).



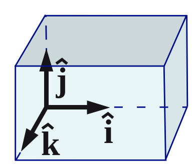
- Complete the following statement with one equal sign $\boxed{=}$ and one not-equal sign $\boxed{\neq}$.

$$\widehat{\mathbf{a}}_x + 2\,\widehat{\mathbf{a}}_y + 3\,\widehat{\mathbf{a}}_z \;\boxed{\quad}\; [\widehat{\mathbf{a}}_x \;\; \widehat{\mathbf{a}}_y \;\; \widehat{\mathbf{a}}_z] * \begin{bmatrix}1\\2\\3\end{bmatrix} \;\boxed{\quad}\; \begin{bmatrix}1\\2\\3\end{bmatrix}$$

2.4 ♣ Fast orthogonal dot-product formula. (Sections 2.9, 2.9.4)

Given: Vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ expressed in terms of right-handed orthogonal unit vectors $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, $\widehat{\mathbf{k}}$, as:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \underbrace{(a_x\,\widehat{\mathbf{i}} + a_y\,\widehat{\mathbf{j}} + a_z\,\widehat{\mathbf{k}})}_{\vec{\mathbf{a}}} \cdot \underbrace{(b_x\,\widehat{\mathbf{i}} + b_y\,\widehat{\mathbf{j}} + b_z\,\widehat{\mathbf{k}})}_{\vec{\mathbf{b}}}$$



- Use the ***distributive property*** for dot products to write $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ in terms of $\widehat{\mathbf{i}} \cdot \widehat{\mathbf{i}}$, $\widehat{\mathbf{i}} \cdot \widehat{\mathbf{j}}$, etc.
- Next, use the ***definition*** of the dot product to calculate $\widehat{\mathbf{i}} \cdot \widehat{\mathbf{i}}$, $\widehat{\mathbf{i}} \cdot \widehat{\mathbf{j}}$, etc. (below-right).
- Simplify $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ to form the ***fast orthogonal dot-product formula***.

Result:
$$\begin{aligned}\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \; & a_x b_x\, \widehat{\mathbf{i}} \cdot \widehat{\mathbf{i}} + a_x b_y\, \widehat{\mathbf{i}} \cdot \widehat{\mathbf{j}} + \boxed{\quad}\, \widehat{\mathbf{i}} \cdot \widehat{\mathbf{k}} \\ + \; & a_y b_x\, \widehat{\mathbf{j}} \cdot \widehat{\mathbf{i}} + a_y b_y\, \boxed{\;} \cdot \boxed{\;} + \boxed{\quad}\, \boxed{\;} \cdot \boxed{\;} \\ + \; & a_z b_x\, \widehat{\mathbf{k}} \cdot \widehat{\mathbf{i}} + a_z b_y\, \boxed{\;} \cdot \boxed{\;} + \boxed{\quad}\, \boxed{\;} \cdot \boxed{\;}\end{aligned}$$

$\widehat{\mathbf{i}} \cdot \widehat{\mathbf{i}} = 1$	$\widehat{\mathbf{i}} \cdot \widehat{\mathbf{j}} = \boxed{\;}$	$\widehat{\mathbf{i}} \cdot \widehat{\mathbf{k}} = \boxed{\;}$
$\widehat{\mathbf{j}} \cdot \widehat{\mathbf{i}} = 0$	$\widehat{\mathbf{j}} \cdot \widehat{\mathbf{j}} = \boxed{\;}$	$\widehat{\mathbf{j}} \cdot \widehat{\mathbf{k}} = \boxed{\;}$
$\widehat{\mathbf{k}} \cdot \widehat{\mathbf{i}} = 0$	$\widehat{\mathbf{k}} \cdot \widehat{\mathbf{j}} = \boxed{\;}$	$\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}} = \boxed{\;}$

$$\boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y + \boxed{\quad}}$$

Use this ***fast orthogonal dot-product formula*** to calculate dot-products when $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, $\widehat{\mathbf{k}}$ are **orthogonal unit** vectors.

Given

$\vec{\mathbf{p}} = 2\,\widehat{\mathbf{i}} + 3\,\widehat{\mathbf{j}} + 4\,\widehat{\mathbf{k}}$
$\vec{\mathbf{q}} = x\,\widehat{\mathbf{i}} + y\,\widehat{\mathbf{j}} + z\,\widehat{\mathbf{k}}$
$\vec{\mathbf{r}} = 5\,\widehat{\mathbf{i}} - 6\,\widehat{\mathbf{j}} + 7\,\widehat{\mathbf{k}}$

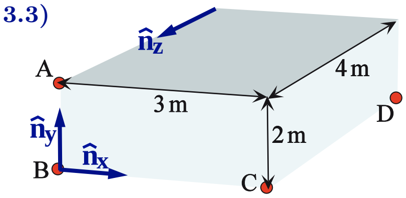
Calculate with the fast orthogonal dot-product formula

$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 2\,x + 3\,y + \boxed{\;}\,z$	$\vec{\mathbf{p}} \cdot \vec{\mathbf{p}} = 29$	$\lvert\vec{\mathbf{p}}\rvert = \sqrt{29}$
$\vec{\mathbf{p}} \cdot \vec{\mathbf{r}} = \boxed{\;}$	$\vec{\mathbf{q}} \cdot \vec{\mathbf{q}} = x^2 + \boxed{\;} + \boxed{\;}$	$\lvert\vec{\mathbf{q}}\rvert = \sqrt{\boxed{\quad}}$
$\vec{\mathbf{q}} \cdot \vec{\mathbf{r}} = \boxed{\quad}$	$\vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = \boxed{\;}$	$\lvert\vec{\mathbf{r}}\rvert = \sqrt{110}$

2.5 ♣ **Perpendicular vectors ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are orthogonal unit vectors). (Section 2.9)**

Draw two vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ that are perpendicular. Hence, $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \square$.
When $\vec{\mathbf{v}} = x\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}$ is perpendicular to $\vec{\mathbf{w}} = 4\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}$, $x = \square$.

2.6 **Dot products to calculate distance and angles. (Sections 2.9, 3.3)**



The figure to the right shows a block with sides of length 2 m, 3 m, 4 m and points A, B, C located at corners. Right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ are directed with $\hat{\mathbf{n}}_x$ from B to C and $\hat{\mathbf{n}}_y$ from B to A.

- Express $\vec{\mathbf{r}}$ (position from A to C) in terms of $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and calculate a numerical value for $|\vec{\mathbf{r}}|^2$. Next, calculate the distance d between A to C (magnitude of $\vec{\mathbf{r}}$).

Result: $\vec{\mathbf{r}} = \square\,\hat{\mathbf{n}}_x - \square\,\hat{\mathbf{n}}_y$ $\qquad |\vec{\mathbf{r}}|^2 \underset{(2.4)}{=} \vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = \square\ \text{m}^2$ $\qquad d = \sqrt{\square}\ \text{m}$

- Calculate the unit vector $\hat{\mathbf{u}}$ directed from A to C and the unit vector $\hat{\mathbf{v}}$ directed from A to D.

Result: $\hat{\mathbf{u}} = \dfrac{3\,\hat{\mathbf{n}}_x - \square\,\hat{\mathbf{n}}_y}{\sqrt{\square}}$ $\qquad \hat{\mathbf{v}} = \dfrac{\square\,\hat{\mathbf{n}}_x - \square\,\hat{\mathbf{n}}_y - \square\,\hat{\mathbf{n}}_z}{\sqrt{\square}}$

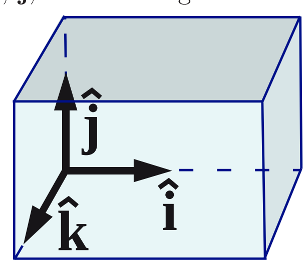
- Calculate $\angle BAC$ (angle between lines $\overline{\text{AB}}$ and $\overline{\text{AC}}$) and $\angle CAD$ (angle between lines $\overline{\text{AC}}$ and $\overline{\text{AD}}$).

Result: $\angle BAC = \square^\circ$ $\qquad \angle CAD = 47.97^\circ$

2.7 ♣ **Construct a unit vector $\hat{\mathbf{u}}$ in the direction of each vector given below. (Section 2.9.2)**

Vector	Unit vector $\hat{\mathbf{u}}$
$3\,\hat{\mathbf{i}}$	$\hat{\mathbf{i}}$
$-3\,\hat{\mathbf{i}}$	$\square$
$3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}$	$\dfrac{\square - \square}{\square}$
$3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 12\,\hat{\mathbf{k}}$	$\dfrac{\square}{\square}$
$c\,\hat{\mathbf{i}}$ c is a real non-zero number	$\dfrac{c}{\square}\,\hat{\mathbf{i}}$ or $\text{sign}(c)\,\hat{\mathbf{i}}$

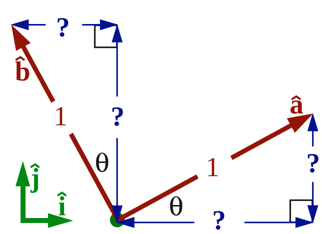
Note: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are orthogonal unit vectors.



Ensure your last answer agrees with your first two answers, e.g., if $c = 3$ or $c = -3$.

2.8 ♣ **Vector components: Sine and cosine. (Section 1.4)**

- **Replace** each **?** in the figure to the right with $\sin(\theta)$ or $\cos(\theta)$.
- Use vector addition to express $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ in terms of $\sin(\theta)$, $\cos(\theta)$, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$.



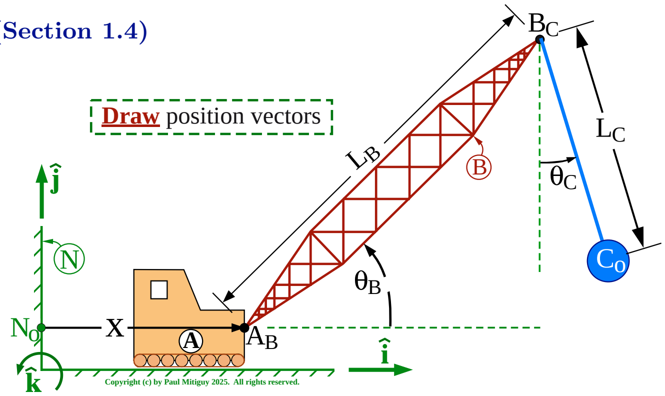
Result: $\hat{\mathbf{a}} = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}}$

Reminder: **SohCahToa** $\qquad \hat{\mathbf{b}} = \square\,\hat{\mathbf{i}} + \cos(\theta)\,\hat{\mathbf{j}}$

2.9 ♣ Vector components for a crane-boom. (Section 1.4)

Shown right is a crane whose cab A supports a boom B that swings a wrecking ball C_o.

Right-handed orthogonal unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are directed with $\hat{\mathbf{i}}$ horizontally-right, $\hat{\mathbf{j}}$ vertically-upward, and $\hat{\mathbf{k}}$ outward-normal to the plane containing points N_o, A_B, B_C, C_o.



Draw each position vector listed below. Then use your knowledge of sine/cosine to resolve these vectors into $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components.

Position from N_o to A_B $\quad {}^{N_o}\vec{r}^{\,A_B} = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}}$

Position from A_B to B_C $\quad {}^{A_B}\vec{r}^{\,B_C} = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}}$

Position from B_C to C_o $\quad {}^{B_C}\vec{r}^{\,C_o} = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}}$

Position from N_o to B_C $\quad {}^{N_o}\vec{r}^{\,B_C} = [\square + \square]\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}}$

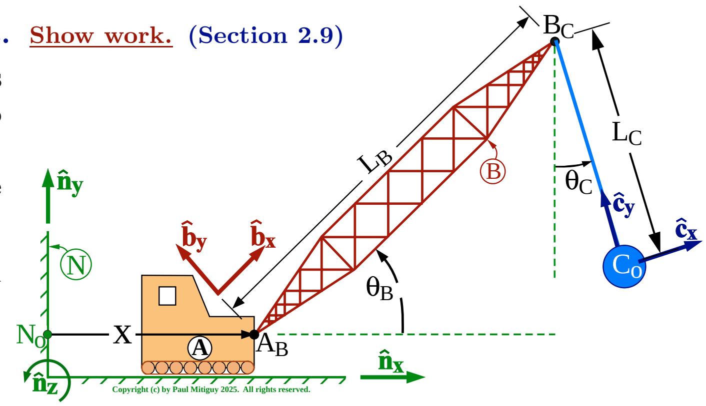
Position from N_o to C_o $\quad {}^{N_o}\vec{r}^{\,C_o} = [\square]\,\hat{\mathbf{i}} + [L_B \sin(\theta_B) - L_C \cos(\theta_C)]\,\hat{\mathbf{j}}$

2.10 Dot products and distance calculations. Show work. (Section 2.9)

Shown right is a crane whose cab A supports a boom B that swings a wrecking ball C_o. To prevent the wrecking ball from hitting a car, the distance between N_o and point B_C (the tip of the boom) must be controlled.

To start this problem, express $\vec{r}$ (the position vector from N_o to B_C) in terms of x, L_B, $\hat{\mathbf{n}}_x$, $\hat{\mathbf{b}}_x$.

Result: $\vec{r} = \square\,\hat{\mathbf{n}}_x + \square\,\hat{\mathbf{b}}_x$



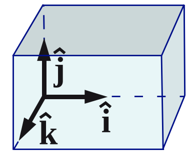
- ***Without*** resolving $\vec{r}$ into $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$ components, use $|\vec{r}| = \sqrt{\vec{r} \cdot \vec{r}}$ [from equation (3.1)] and the distributive property to calculate the distance between N_o and B_C in terms of x, L_B, θ_B.
Result: (if stumped, hint below).[1] **Optional:** Calculate $|\vec{r}|$ when $x = 20$ m, $L_B = 10$ m, $\theta_B = 30°$.

$$\text{Distance between } N_o \text{ and } B_C: \quad |\vec{r}| = \sqrt{\square^2 + \square^2 + 2\,x\,L_B \cos(\theta_B)} \approx 29.1 \text{ m}$$

- Homework 2.9 showed $\vec{r}$ can be expressed as $\vec{r} = [x + L_B \cos(\theta_B)]\,\hat{\mathbf{n}}_x + L_B \sin(\theta_B)\,\hat{\mathbf{n}}_y$.
Use this expression to verify your previous result for $|\vec{r}| = \sqrt{\vec{r} \cdot \vec{r}}$.
Result: $|\vec{r}|$ simplifies to the previous result but uses an inefficient process and $\sin^2(\theta_B) + \cos^2(\theta_B) = 1$.

[1] Hint: The distributive property for vector dot-multiplication is $(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$. Use the distributive property to express $\vec{r} \cdot \vec{r}$ in terms of x, L_B, and $\hat{\mathbf{n}}_x \cdot \hat{\mathbf{b}}_x$. Thereafter, use the ***dot-product definition*** of $(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{b}}_x)$ to form $\vec{r} \cdot \vec{r} = \square^2 + \square^2 + 2\,x\,L_B\,(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{b}}_x) \underset{(2.2)}{=} \square^2 + \square^2 + 2\,x\,L_B \cos(\square)$.

2.11 ♣ **Cross products with right-handed orthogonal unit vectors.** **(Section 2.10)**



Given: Vectors $\vec{v}$ and $\vec{w}$ expressed in terms of right-handed orthogonal unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$, with: $\vec{v} \times \vec{w} = \underbrace{(a\,\hat{i} + b\,\hat{j} + c\,\hat{k})}_{\vec{v}} \times \underbrace{(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})}_{\vec{w}}$

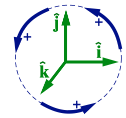
- Use the ***distributive property*** for cross products to write $\vec{v} \times \vec{w}$ in terms of $\hat{i} \times \hat{i}$, $\hat{i} \times \hat{j}$, etc. Next, use the ***definition*** of the cross product to calculate $\hat{i} \times \hat{i}$, $\hat{i} \times \hat{j}$, etc. (below-right).

Result:

$$\begin{aligned}\vec{v} \times \vec{w} = \; & a\,x\;\hat{i} \times \hat{i} \;+\; a\,y\;\hat{i} \times \hat{j} \;+\; \square\;\hat{i} \times \hat{k}\\ +\; & b\,x\;\hat{j} \times \hat{i} \;+\; b\,y\;\square \times \square \;+\; \square\;\square \times \square\\ +\; & c\,x\;\hat{k} \times \hat{i} \;+\; c\,y\;\square \times \square \;+\; \square\;\square \times \square\end{aligned}$$

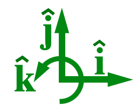
$\hat{i} \times \hat{i} = \vec{0}$	$\hat{i} \times \hat{j} = \hat{k}$	$\hat{i} \times \hat{k} = -\hat{j}$
$\hat{j} \times \hat{i} = \square$	$\hat{j} \times \hat{j} = \square$	$\hat{j} \times \hat{k} = \square$
$\hat{k} \times \hat{i} = \square$	$\hat{k} \times \hat{j} = \square$	$\hat{k} \times \hat{k} = \square$

- Combine your previous results to calculate $\vec{v} \times \vec{w}$ in terms of a, b, c, x, y, z.



Result: $\vec{v} \times \vec{w} = (b\,z - \square)\,\hat{i} + (\square - a\,z)\,\hat{j} + (\square)\,\hat{k}$

2.12 ♣ **Cross products and determinants (orthogonal unit vectors).** **(Section 2.10.2)**



Shown right are arbitrary vectors $\vec{v}$ and $\vec{w}$ expressed in terms of right-handed orthogonal unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$. Show that calculating $\vec{v} \times \vec{w}$ with the ***distributive property*** of the cross product (seen in Hw 2.11) happens to be equal to the ***determinant*** of the matrix shown to the right.

$$\vec{v} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$$
$$\vec{w} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$
$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{bmatrix}$$

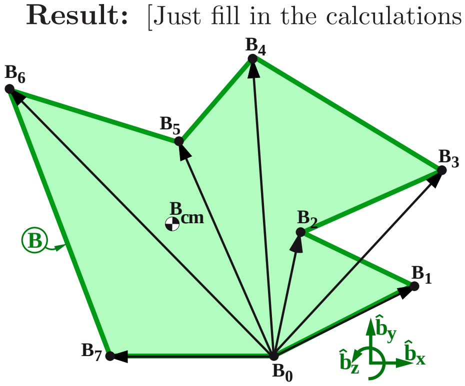
Result: $\vec{v} \times \vec{w} = (b\,z - \square)\,\hat{i} + (\square - a\,z)\,\hat{j} + (\square)\,\hat{k}$

2.13 ♣ **Cross products: Commercial area calculation algorithm (surveying).** **(Sections 2.10.1, 3.2)**

Complex **planar objects** such as the polygon B below can be decomposed into triangles for important planar measurements (e.g., farming acreage, building costs, and mass and area properties of 2D objects).

- Calculate $\vec{A}_2$ and $\vec{A}_4$, the vector-areas of triangles $B_0\,B_2\,B_3$ and $B_0\,B_4\,B_5$.
- Account for overlapped areas with **positive** and **negative** vector areas.

Result: [Just fill in the calculations for $\vec{A}_2$, $\vec{A}_4$, and $\mathbf{A}$ using eqn (3.3)].

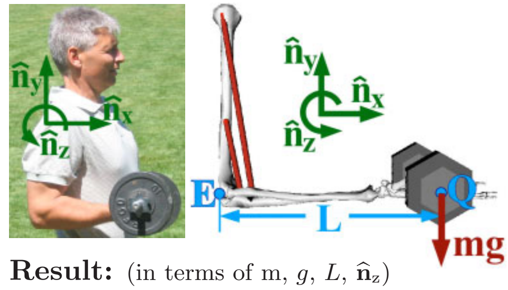


$$\begin{aligned}
\vec{r}_1 &= {}^{B_0}\vec{r}^{\,B_1} = 2.0\,\hat{b}_x + 2.0\,\hat{b}_y\\
\vec{r}_2 &= {}^{B_0}\vec{r}^{\,B_2} = 0.5\,\hat{b}_x + 2.5\,\hat{b}_y\\
\vec{r}_3 &= {}^{B_0}\vec{r}^{\,B_3} = 3.0\,\hat{b}_x + 4.0\,\hat{b}_y\\
\vec{r}_4 &= {}^{B_0}\vec{r}^{\,B_4} = -0.5\,\hat{b}_x + 7.0\,\hat{b}_y\\
\vec{r}_5 &= {}^{B_0}\vec{r}^{\,B_5} = -1.0\,\hat{b}_x + 5.0\,\hat{b}_y\\
\vec{r}_6 &= {}^{B_0}\vec{r}^{\,B_6} = -3.0\,\hat{b}_x + 6.0\,\hat{b}_y\\
\vec{r}_8 &= {}^{B_0}\vec{r}^{\,B_8} = -2.0\,\hat{b}_x + 0.0\,\hat{b}_y
\end{aligned}$$

$$\begin{aligned}
\vec{A}_1 &= \tfrac{1}{2}\,(\vec{r}_1 \times \vec{r}_2) = 2\,\hat{b}_z\\
\vec{A}_2 &= \tfrac{1}{2}\,(\vec{r}_2 \times \vec{r}_3) = \square.75\,\hat{b}_z\\
\vec{A}_3 &= \ldots = 11.5\,\hat{b}_z\\
\vec{A}_4 &= \ldots = \square.25\,\hat{b}_z\\
\vec{A}_5 &= \ldots = 4.5\,\hat{b}_z\\
\vec{A}_6 &= \tfrac{1}{2}\,(\vec{r}_6 \times \vec{r}_7) = 6\,\hat{b}_z\\
\vec{\mathbf{A}} &= \textstyle\sum_{i=1}^{6} \vec{A}_i = \square\\
\text{Area} &= |\vec{\mathbf{A}}| = \vec{\mathbf{A}} \cdot \hat{b}_z = 23.5
\end{aligned}$$

2.14 Biomechanics: Gravity moment for curling $\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ (Section 2.10)

The figures to the right show an athlete curling a dumbbell (modeled as a particle Q of mass m). The forearm connects to the upper arm at the elbow (point E). Orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are directed with $\widehat{\mathbf{n}}_x$ from E to Q and $\widehat{\mathbf{n}}_y$ vertically upward.



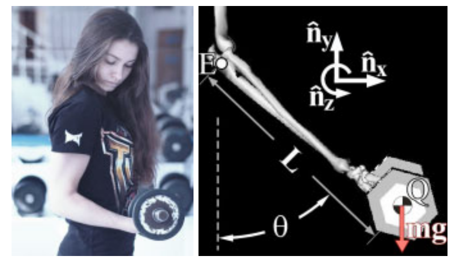
Description	Symbol	Type
Earth's gravitational constant	g	$g \approx 9.8 \frac{\text{m}}{\text{s}^2}$
Mass of dumbbell Q	m	Positive constant
Distance between elbow E and Q	L	Positive constant

Determine the moment of gravity forces on Q about E as $\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ where $\vec{\mathbf{r}} = L\,\widehat{\mathbf{n}}_x$ and $\vec{\mathbf{F}} = \text{-m}\,g\,\widehat{\mathbf{n}}_y$.

Result: (in terms of m, g, L, $\widehat{\mathbf{n}}_z$)

$\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ = ______

Now consider the forearm making an angle θ with downward vertical. Form $\vec{\mathbf{M}}$ and its magnitude $|\vec{\mathbf{M}}|$. Determine the values of θ $(0 \le \theta \le 180°)$ that produce maximum and minimum $|\vec{\mathbf{M}}|$.
To simplify $|\vec{\mathbf{M}}|$, note m, g, L are positive and for $0 \le \theta \le 180°$, $\sin(\theta) \ge 0$.



Result: (in terms of m, g, L, θ, $\widehat{\mathbf{n}}_z$).

$\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ = ______ $\qquad |\vec{\mathbf{M}}|$ = ______

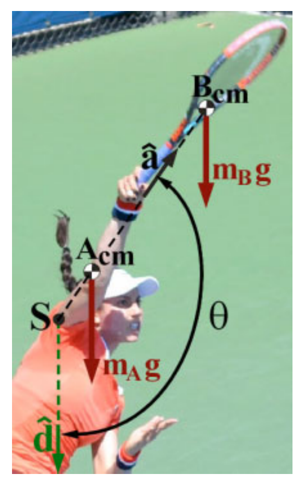
Max $|\vec{\mathbf{M}}|$ = ______ at θ = ______°

Min $|\vec{\mathbf{M}}|$ = ______ at θ = ______° or ______°

Optional: Modeling the elbow as a revolute joint, draw a ***free-body diagram (FBD)*** of the system consisting of the forearm and dumbbell.

2.15 Biomechanics: Gravity force and moment for tennis $\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ (Section 2.10)

Shown right is an athlete whose arm A swings a tennis racquet B. Point S (shoulder), A_{cm} (A's center of mass), and B_{cm} (B's center of mass) lie along a line parallel to a unit vector $\widehat{\mathbf{a}}$. The unit vector $\widehat{\mathbf{d}}$ is vertically-downward ↓.



Description	Symbol	Type
Earth's gravitational constant	g	$g \approx 9.8 \frac{\text{m}}{\text{s}^2}$
Mass of A, mass of B	m_A, m_B	Positive constants
Distances between S and A_{cm} and S and B_{cm}	L_A, L_B	Positive constants
Angle between $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{d}}$	θ	$0 \le \theta \le 180°$

- Form $\vec{\mathbf{F}}_{\text{gravity}}$ (the net force on A and B due to Earth's gravity).
- Form $|\vec{\mathbf{M}}|$ (the magnitude of the moment of those gravity forces about S).
  Note: $\vec{\mathbf{M}} = {}^{S}\vec{\mathbf{r}}^{A_{cm}} \times m_A\,g\,\widehat{\mathbf{d}} \;+\; {}^{S}\vec{\mathbf{r}}^{B_{cm}} \times m_B\,g\,\widehat{\mathbf{d}}$.

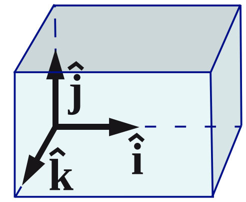
Result: $\vec{\mathbf{F}}_{\text{gravity}}$ = (______) $\widehat{\mathbf{d}}$

$|\vec{\mathbf{M}}|$ = ______

Optional: Modeling the athlete grip of the racquet as a weld, draw a ***free-body diagram (FBD)*** of the racquet. Next, choose a model for the shoulder joint and draw a ***FBD*** of the system consisting of the arm and racquet.

2.16 Scalar triple product with bases (Section 2.11).

The figure shows right-handed orthogonal unit vectors $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, $\widehat{\mathbf{k}}$.



Given	Calculate
$\vec{\mathbf{u}} = 2\,\widehat{\mathbf{i}} + 3\,\widehat{\mathbf{j}} + 4\,\widehat{\mathbf{k}}$	$\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{u}} = \square$
$\vec{\mathbf{v}} = x\,\widehat{\mathbf{i}} + y\,\widehat{\mathbf{j}} + z\,\widehat{\mathbf{k}}$	$\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \square\, z - \square\, x - 6\,y$
$\vec{\mathbf{w}} = 5\,\widehat{\mathbf{i}} - 6\,\widehat{\mathbf{j}} + 7\,\widehat{\mathbf{k}}$	$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{w}} = \square\, z - 45\,x - \square\, y$

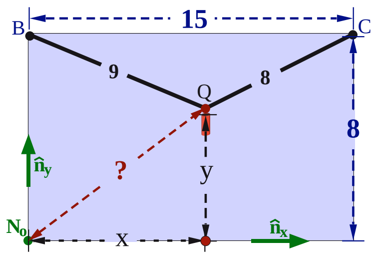
Note: There is a unique order of operations in $\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$, but parentheses clarify your work.

- $\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{w}}$ and it is OK to switch $\cdot$ and $\times$ in scalar triple products. **True/False**

.

2.17 Locating a microphone (2D). Show work. (Section 1.4)

A microphone Q is attached to two pegs B and C by two cables. Knowing the peg locations, cable lengths, and points B, C, Q, N_o all lie in the same plane, determine the distance between Q and N_o. Do the problem with Euclidean geometry (e.g., law of cosines), then try vectors (see Hw 1.34).

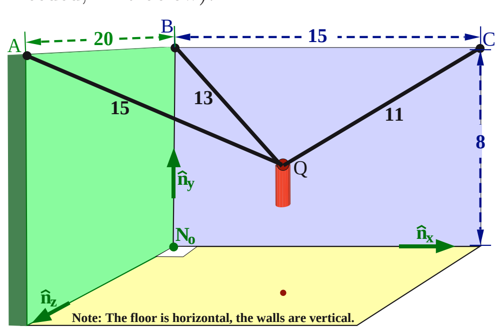


Distance between B to C		15 m
Distance between N_o to B	h	8 m
Length of cable joining B and Q	L_B	9 m
Length of cable joining C and Q	L_C	8 m
Distance between N_o and Q		9.01 m

Although there are two mathematical answers to this problem, one is above the ceiling by ≈ 12 m and requires the cables to be in compression.

2.18 †Locating a microphone (3D).

A microphone Q is attached to three pegs A, B, C by three cables. Knowing the peg locations, cable lengths, and the walls are orthogonal, determine the distance between Q and point N_o. Show work. (If needed, hint below).[2]

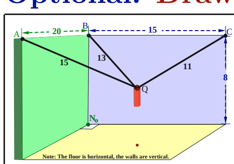
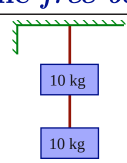
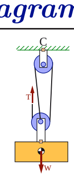
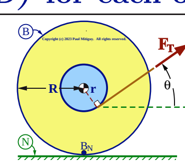
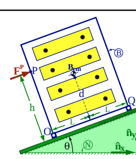


Distance between A to B		20 m
Distance between B to C		15 m
Distance between N_o to B	h	8 m
Length of cable joining A and Q	L_A	15 m
Length of cable joining B and Q	L_B	13 m
Length of cable joining C and Q	L_C	11 m
Distance between N_o and Q If Q is above ceiling, distance ≈ 17 m		13.3 m

Note: This is part of the process of a camera targeting a football/baseball in a stadium or a laser targeting cancer or ...

Vocabulary: In this ***forward kinematics*** analysis, the knowns cable lengths determine the position of "end-effector" Q.

2.19 Optional: Draw the *free-body diagram (FBD)* for each object below.

					
Particle Q Hw 2.18	**Top block** Hw 12.13	**Bottom pulley** Hw 12.15	**Rolling spool** B Hw 13.14	**Bureau** B Hw 13.10	**Entire system** Hw 11.19

[2]Hint: See Hw 1.34 or Section 3.3. Introduce unknowns x, y, z so Q's position from N_o is $x\,\widehat{\mathbf{n}}_x + y\,\widehat{\mathbf{n}}_y + z\,\widehat{\mathbf{n}}_z$. Although nonlinear equations are usually solved with a computer, these can be solved "by-hand". Or, go to www.WolframAlpha.com and type

Solve `x^2 + (-20+z)^2 + (-8+y)^2 = 225,   x^2 + z^2 + (-8+y)^2 = 169,   z^2 + (-15+x)^2 + (-8+y)^2 = 121`