

Basis independent vector operations: $-\vec{b}$ $5\vec{b}$ $\vec{a} + \vec{b}$ $\angle(\vec{a}, \vec{b})$ $\vec{a} \cdot \vec{b}$ $\vec{a} \times \vec{b}$

Show work – except for ♣ fill-in-blanks (print .pdf from www.MotionGenesis.com ⇒ [Textbooks](#) ⇒ [Resources](#)).

1.1 ♣ Solving problems – what physicists and engineers do.

Understanding dynamics results from **doing** problems. Many problems herein guide you to help you synthesize processes (imitation). Please **do** these problems by yourself or with colleagues/instructors and use the textbook and other resources.



Confucius 500 B.C.

“I hear and I forget.

I see and I remember.

I and I understand.”

“By three methods we may learn wisdom:

1st by reflection, which is noblest;

2nd by imitation, which is easiest;

3rd by experience, which is the bitterest.”

1.2 ♣ What is a vector (as defined by Gibbs circa 1897)? (Section 2.2)

Two properties (attributes) of a vector are and (fill in the blanks).

1.3 ♣ What is a zero vector? (Section 2.3)

A zero vector $\vec{0}$ has a magnitude of 0 ($|\vec{0}| = 0$). True/False (circle true or false).

A zero vector $\vec{0}$ has a direction. True/False

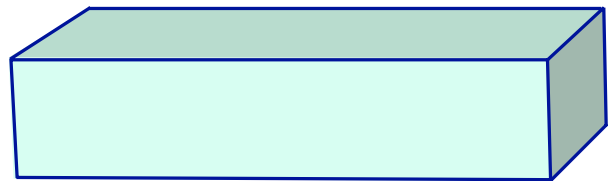
anyVector + $\vec{0}$ = anyVector True/False

1.4 ♣ Unit vectors. (Section 2.4)

All unit vectors have a magnitude of 1 (e.g., $ \hat{i} = 1$, $ \hat{j} = 1$, $ \hat{k} = 1$).	True/False
Typically, a unit vector is denoted with a hat, e.g., as \hat{k} rather than \vec{k} .	True/False
All unit vectors are equal.	True/False
A unit vector \hat{u} in the direction of the non-zero vector \vec{v} is $\hat{u} = \frac{\vec{v}}{ \vec{v} }$.	True/False

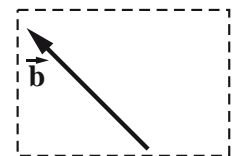
1.5 ♣ Draw the vectors \vec{a} , \vec{b} , \hat{c} , \hat{d} (Section 2.2)

- \vec{a} Horizontally-right vector.
- \vec{b} Vertically-upward vector.
- \hat{c} Outwardly-directed **unit** vector.
- \hat{d} Inwardly-directed **unit** vector.



1.6 ♣ Negating a vector. (Section 2.8)

Draw the vector $-\vec{b}$. Negating the vector \vec{b} results in a vector with different:
 magnitude direction orientation sense (circle **all** that apply)

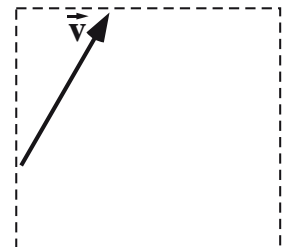


Historical note: Negative numbers (e.g., -3) were not widely accepted until 1800 A.D.

1.7 ♣ Vector magnitude and direction (orientation and sense). (Section 2.2)

The figure to the right shows a vector \vec{v} . **Draw** the vectors \vec{a} , \vec{b} , \vec{c} , \vec{d} , \vec{e} .

- | | |
|-----------|--|
| \vec{a} | Same magnitude and direction as \vec{v} ($\vec{a} = \vec{v}$). |
| \vec{b} | Same magnitude as \vec{v} , with $\vec{b} = -\vec{v}$ (antiparallel). |
| \vec{c} | Same magnitude as \vec{v} , but different direction with $\vec{c} \neq -\vec{v}$. |
| \vec{d} | Smaller magnitude than \vec{v} , but same direction as \vec{v} . |
| \vec{e} | Different magnitude and different direction than \vec{v} . |



1.8 ♣ **Vector magnitude and direction** (Section 2.2)

Knowing x is a real number (e.g., -3 or 0 or 7.8) and \hat{u} is a horizontal unit vector \rightarrow , complete **magnitude** with \leq or \geq and complete **direction** with $+\hat{u}$ or $-\hat{u}$.

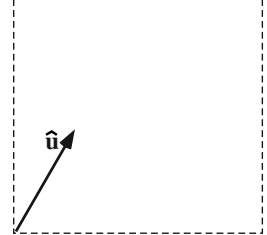
Vector	with	Magnitude	Direction
$x\hat{u}$	$x \geq 0$	$ x\hat{u} \geq 0$	$+\hat{u}$
$x\hat{u}$	$x \leq 0$	$ x\hat{u} \geq 0$	
$-x\hat{u}$	$x \geq 0$	$ x\hat{u} \geq 0$	
$-x\hat{u}$	$x \leq 0$	$ x\hat{u} \geq 0$	

1.9 ♣ **Multiplying a vector by a scalar.** (Section 2.7)

The following statements involve a unit vector \hat{u} and a real scalar s ($s \neq 0$). If a statement is true, provide a numerical value for s that supports your answer

- $s\hat{u}$ can have a different **magnitude** than \hat{u} . True/False $s =$
- $s\hat{u}$ can have a different **direction** than \hat{u} . True/False $s =$
- $s\hat{u}$ can have a different **sense** than \hat{u} . True/False $s =$
- $s\hat{u}$ can have a different **orientation** than \hat{u} . True/False $s =$

Draw $2\hat{u}$ and $-2\hat{u}$.



1.10 ♣ **Graphical vector addition/subtraction.** (Sections 2.6, 2.8)

Draw $\vec{a} + \vec{b}$

Draw $\vec{b} + \vec{a}$

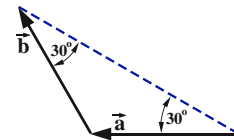
Draw $\vec{a} - \vec{b}$

Draw $\vec{b} - \vec{a}$

1.11 ♣ **Angle $\angle(\vec{a}, \vec{b})$ between vectors.** (Section 2.9)

For the figure shown right, determine the numerical value for the angle between vector \vec{a} and vector \vec{b} .

Result: $\angle(\vec{a}, \vec{b}) =$ °



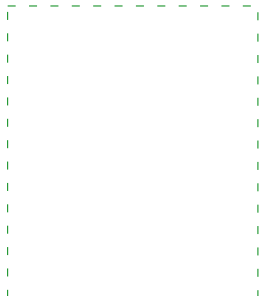
1.12 ♣ **Visual representation of a vector dot-product.** (Section 2.9)

Write the **definition** of the dot-product of a vector \vec{a} with a vector \vec{b} . Include a **sketch** with **each symbol** in your definition clearly labeled.

Result: $\vec{a} \cdot \vec{b} \triangleq$

Knowing \vec{a} and \vec{b} are arbitrary vectors, complete the blanks with \leq , $=$, or \geq .

When the angle between \vec{a} and \vec{b} is 0°	$\vec{a} \cdot \vec{b}$ <input type="text"/> 0	(parallel)
When the angle between \vec{a} and \vec{b} is 90°	$\vec{a} \cdot \vec{b}$ <input type="text"/> 0	(perpendicular)
When the angle between \vec{a} and \vec{b} is 180°	$\vec{a} \cdot \vec{b}$ <input type="text"/> 0	(antiparallel)
For arbitrary vectors \vec{a} and \vec{b} ,	$\vec{a} \cdot \vec{b}$ <input type="text"/> $\vec{b} \cdot \vec{a}$	



Sketch should include \vec{a} , \vec{b} , $|\vec{a}|$, $|\vec{b}|$, θ .

1.13 ♣ **Visual representation of a vector cross-product.** (Section 2.10)

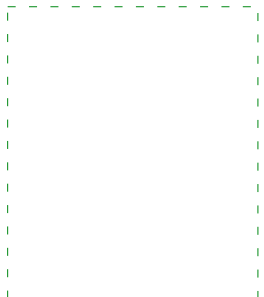
Write the **definition** of the cross-product of a vector \vec{a} with a vector \vec{b} . Include a **sketch** with **each symbol** in your definition clearly labeled.

Result: $\vec{a} \times \vec{b} \triangleq$ $(\theta) \hat{u}$

where \hat{u} is
 and θ is

Knowing \vec{a} and \vec{b} are non-zero vectors, complete the blanks with $=$ or \neq .

When the angle between \vec{a} and \vec{b} is 0°	$\vec{a} \times \vec{b}$ <input type="text"/> $\vec{0}$	(parallel)
When the angle between \vec{a} and \vec{b} is 90°	$\vec{a} \times \vec{b}$ <input type="text"/> $\vec{0}$	(perpendicular)
When the angle between \vec{a} and \vec{b} is 180°	$\vec{a} \times \vec{b}$ <input type="text"/> $\vec{0}$	(antiparallel)
For arbitrary vectors \vec{a} and \vec{b} ,	$\vec{a} \times \vec{b}$ <input type="text"/> $\vec{b} \times \vec{a}$	



Sketch should include \vec{a} , \vec{b} , $|\vec{a}|$, $|\vec{b}|$, θ , \hat{u} .

1.14 ♣ **Properties of vector dot-products and cross-products.** Show work. (Sections 2.9.1, 2.10)

When \vec{a} is <i>parallel</i> to \vec{b} ,	$\vec{a} \cdot \vec{b} = 0$	True/False	$\vec{a} \times \vec{b} = \vec{0}$	True/False
When \vec{a} is <i>perpendicular</i> to \vec{b} ,	$\vec{a} \cdot \vec{b} = 0$	True/False	$\vec{a} \times \vec{b} = \vec{0}$	True/False
For arbitrary vectors \vec{a} and \vec{b} ,	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	True/False	$\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$	True/False

1.15 **Dot-products and cross-products via definitions.** Show work. (Sections 2.9, 2.10)

Redraw each figure to clarify $\angle(\vec{a}, \vec{b})$, the angle between \vec{a} and \vec{b} (useful for dot and cross-product). On each redrawn figure, **draw** a unit vector \hat{k} outward-normal to the plane of the paper (perpendicular to \vec{a} and \vec{b}). Next, knowing $|\vec{a}| = 2$ and $|\vec{b}| = 4$, calculate the expressions below (2^+ significant digits) using only the definitions of dot-product and cross-product.

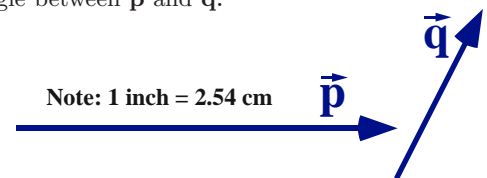
$\angle(\vec{a}, \vec{b}) = \square^\circ$ $\angle(\vec{a}, \vec{b}) = \square^\circ$ $\angle(\vec{a}, \vec{b}) = \square^\circ$ $\angle(\vec{a}, \vec{b}) = \square^\circ$
 $\vec{a} \cdot \vec{b} = \square$ $\vec{a} \cdot \vec{b} = \square$ $\vec{a} \cdot \vec{b} = \square$ $\vec{a} \cdot \vec{b} = \square$
 $\vec{a} \times \vec{b} = \square$ $\vec{a} \times \vec{b} = \square$ $\vec{a} \times \vec{b} = \square$ $\vec{a} \times \vec{b} = \square$

1.16 **Visual estimation of vector dot/cross-products.** Show work. (Sections 2.9, 2.10)

Estimate the magnitude of the vector \vec{q} shown below, the angle between \vec{p} and \vec{q} , $\vec{p} \cdot \vec{q}$, and the magnitude of $\vec{p} \times \vec{q}$. **Show work** and **redraw** to clarify the angle between \vec{p} and \vec{q} .

Result: (Provide numerical results with 1 or more significant digits).

$ \vec{p} \approx 4.0$ cm	$ \vec{q} \approx \square$ cm	$\angle(\vec{p}, \vec{q}) \approx \square^\circ$
$\vec{p} \cdot \vec{q} \approx \square$ cm ²	$ \vec{p} \times \vec{q} \approx \square$ cm ²	



1.17 ♣ **Vector operations and units.** (Chapter 2)

Each vector operation below involves a position vector \vec{r} (with **units** of m) and/or a velocity vector \vec{v} (with **units** of $\frac{m}{s}$). Determine whether the operation produces a well-defined scalar or vector or is **undefined**. If well-defined, determine the associated units.

Operation:	$-\vec{r}$	$5\vec{v}$	$5\frac{m}{s} + \vec{v}$	$\vec{r} + 2\vec{r}$	$\vec{r} + \vec{v}$	$5\frac{m}{s} \cdot \vec{v}$	$\vec{r} \cdot \vec{v}$	$\vec{r} \times \vec{v}$
Produces:	vector							
Units:	meters							

1.18 ♣ **Vector exponentiation: $\vec{v}^2 = \vec{v} \cdot \vec{v}$ and \vec{v}^3 .** (Section 2.9)

The following is a reasonable proof that $\vec{v}^2 = \vec{v} \cdot \vec{v}$. **True/False** (if False, provide a proof).

$$\vec{v}^2 \triangleq |\vec{v}|^2 \quad \vec{v} \cdot \vec{v} \triangleq_{(2.2)} |\vec{v}| |\vec{v}| \cos(0^\circ) = |\vec{v}|^2 \quad \vec{v}^2 = \vec{v} \cdot \vec{v}$$

Complete the proof that relates \vec{v}^3 to $\vec{v} \cdot \vec{v}$ raised to a real number.

Result: $|\vec{v}| \stackrel{(2.4)}{=} \sqrt{\square \cdot \square}$ $\vec{v}^3 \triangleq |\vec{v}|^{\square} = (\sqrt{\square \cdot \square})^{\square} = (\vec{v} \cdot \vec{v})^{\frac{3}{2}}$

1.19 ♣ **$|c\hat{a}_x|$ Calculate vector magnitude with dot products.** (Section 2.9 and Hw 1.18)

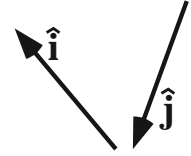
Show how the vector dot-product can be used to show that the magnitude of the vector $c\hat{a}_x$ (c is a positive or **negative** number and \hat{a}_x is a unit vector) can be written solely in terms of c (without \hat{a}_x).

Result: $|c\hat{a}_x| = +\sqrt{\square \cdot \square} = +\sqrt{c^2 * \square \cdot \square} = +\sqrt{c^2} = \text{abs}(c)$

1.20 †(Challenge) **Magnitude of the vector \vec{v} .** Show work. (Section 2.9)

Knowing the angle between a unit vector \hat{i} and unit vector \hat{j} is 120° , calculate a numerical value for the magnitude of $\vec{v} = 3\hat{i} + 4\hat{j}$.

Result: $|\vec{v}| = \sqrt{13}$ Note: The answer is not $\sqrt{25} = 5$.



1.21 ♣ **Property of scalar triple product.** (Section 2.11)

For arbitrary non-zero vectors $\vec{a}, \vec{b}, \vec{c}$: $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ **Never/Sometimes/Always**
 A property of the *scalar triple product* is $\vec{a} \cdot \vec{b} \times \vec{a} = 0$. **True/False.**

1.22 **Property of vector triple cross-product.** (Sections 2.10, 2.11)

Complete the following equation: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\text{yellow}) - \vec{c}(\text{yellow})$

For arbitrary vectors $\vec{a}, \vec{b}, \vec{c}$: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c} + \vec{b} \times (\vec{a} \times \vec{c})$ **True/False** (show work).

1.23 ♣ **Optional: Proof of magnitude of vector cross product property.** (Sections 2.9, 2.10)

Letting $\hat{\lambda}$ be a *unit vector* and \vec{v} be *any vector*, prove¹ $|\vec{v} \times \hat{\lambda}|^2 = \vec{v} \cdot \vec{v} - (\vec{v} \cdot \hat{\lambda})^2$.

1.24 ♣ **Form the unit vector \hat{u} having the same direction as $c\hat{a}_x$.** (Section 2.4)

Result: $\hat{u} = \text{yellow} \hat{a}_x$ Note: \hat{a}_x is a unit vector and c is a non-zero real number, e.g., 3 or -3.

1.25 ♣ **Coefficient of \hat{u} in cross products – definitions and trig functions.** (Section 2.10)

The *cross product* of vectors \vec{a} and \vec{b} can be written in terms of a real scalar s as $\vec{a} \times \vec{b} = s\hat{u}$ where \hat{u} is a unit vector perpendicular to both \vec{a} and \vec{b} in a direction defined by the **right-hand rule**. The coefficient s of the unit vector \hat{u} is inherently non-negative. **True/False.**

1.26 ♣ **Ranges of angles from dot-product and cross-product calculations.** (Sections 2.9, 2.10)

Quantity	Numerical range of values
$c = \hat{a} \cdot \hat{b}$ (assume \hat{a} and \hat{b} are known so a numerical value for $\hat{a} \cdot \hat{b}$ can be calculated).	$\text{yellow} \leq c \leq \text{yellow}$
$s = \hat{a} \times \hat{b} $ (assume \hat{a} and \hat{b} are known so a numerical value for $ \hat{a} \times \hat{b} $ can be calculated).	$\text{yellow} \leq s \leq \text{yellow}$
Angle θ_c between \hat{a} and \hat{b} that can be uniquely determined solely from c .	$\text{yellow}^\circ \leq \theta_c \leq \text{yellow}^\circ$
Angle θ_s between \hat{a} and \hat{b} that can be uniquely determined solely from s .	$\text{yellow}^\circ \leq \theta_s \leq \text{yellow}^\circ$
Angle θ between \hat{a} and \hat{b} , i.e., $\theta = \angle(\hat{a}, \hat{b})$	$\text{yellow}^\circ \leq \theta \leq \text{yellow}^\circ$

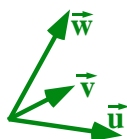
Note: The range of θ_s is smaller than the range for θ . Hence, s and θ_s are insufficient to correctly calculate θ .

1.27 ♣ **Using vector identities to simplify expressions** (refer to Homework 1.14)

One reason to treat vectors as *basis-independent* quantities is to simplify vector expressions **without** resolving the vectors into orthogonal “ $\vec{x}, \vec{y}, \vec{z}$ ” or “ $\vec{i}, \vec{j}, \vec{k}$ ” components. Simplify the following vector expressions using mathematical properties of dot-products and cross-products.

Express results in terms of dot-products \cdot and cross-products \times of the arbitrary vectors $\vec{u}, \vec{v}, \vec{w}$.

$\vec{u}, \vec{v}, \vec{w}$ are not necessarily orthogonal or co-planar.



Vector expression	Simplified vector expression
$(3\vec{u} - 2\vec{v}) \times (\vec{u} + \vec{v})$	$\text{yellow} \vec{u} \times \vec{v}$
$(3\vec{u} - 2\vec{v}) \cdot (\vec{u} + \vec{v})$	$\text{yellow} \vec{u}^2 - \text{yellow} \vec{v}^2 + \text{yellow} \vec{u} \cdot \vec{v}$
$(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v})$	$\text{yellow} - \text{yellow}$
$(3\vec{u} - 2\vec{v}) \times (\vec{u} + \vec{v}) \cdot (2\vec{u} - 7\vec{v})$	yellow
$(\vec{u} + \vec{v}) \times (\vec{v} + 2\vec{w}) \cdot (\vec{w} + 2\vec{u})$	$\text{yellow} \vec{u} \times \vec{v} \cdot \vec{w}$

¹One way to prove this is to write $(\vec{v} \times \hat{\lambda})^2 = (\vec{v} \times \hat{\lambda}) \cdot (\vec{v} \times \hat{\lambda}) \stackrel{(2.10)}{=} \vec{v} \cdot [\hat{\lambda} \times (\vec{v} \times \hat{\lambda})]$ and then use the vector triple cross-product property $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ from Section 2.10. Alternatively, it is helpful to write $\vec{v} = \vec{v}_\perp \hat{\lambda}_\perp + \vec{v}_\parallel \hat{\lambda}$ where $\vec{v}_\perp \hat{\lambda}_\perp$ is the component of \vec{v} that is perpendicular to $\hat{\lambda}$ and $\vec{v}_\parallel \hat{\lambda}$ is the component of \vec{v} that is parallel to $\hat{\lambda}$.

1.28 ♣ **Vector concepts: Solving a vector equation?** (Section 2.9.5)

Shown right is a vector equation and a questionable process that solves for v_x ($\hat{\mathbf{a}}_x$ is a unit vector and $v_x, \dot{\theta}, R$ are scalars).

$$v_x \hat{\mathbf{a}}_x = \dot{\theta} R \hat{\mathbf{a}}_x$$

$$v_x = \dot{\theta} R \frac{\hat{\mathbf{a}}_x}{\hat{\mathbf{a}}_x} = \dot{\theta} R$$

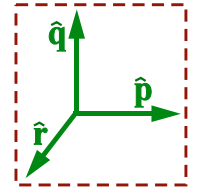
This is a valid process to solve for v_x . **True/False.**

Explain:

1.29 **Changing a vector equation to scalar equations.** (Section 2.9.5)

Shown right are three mutually orthogonal unit vectors $\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{r}}$.

Use a vector operation (e.g., +, *, ·, ×) to change the **vector** equation $(2x-4)\hat{\mathbf{p}} = \vec{\mathbf{0}}$ into **one scalar** equation and subsequently solve the scalar equation for x .



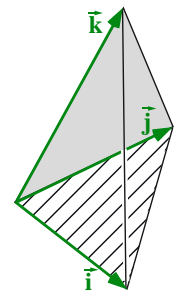
Result: $(2x-4)\hat{\mathbf{p}} = \vec{\mathbf{0}} \quad \overset{??}{\Rightarrow} \quad (2x-4) = 0 \Rightarrow x = 2$

Show **every** vector operation (e.g., +, *, ·, ×) that changes the following **vector** equation into **three scalar** equations and subsequently solve the scalar equations for x, y, z .

$$(2x-4)\hat{\mathbf{p}} + (3y-9)\hat{\mathbf{q}} + (4z-16)\hat{\mathbf{r}} = \vec{\mathbf{0}}$$

Result: $(2x-4) = 0 \quad (3y-9) = 0 \quad (\text{yellow box}) = 0$
 $x = 2 \quad y = 3 \quad z = 4$

†**Optional:** The figure to the right shows three **non-orthogonal**, non-coplanar vectors $\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}$. Show **every** vector operation that changes the following **vector** equation into **three** uncoupled **scalar** equations and subsequently solve those scalar equations for x, y, z .



$$(2x-4)\vec{\mathbf{i}} + (3y-9)\vec{\mathbf{j}} + (4z-16)\vec{\mathbf{k}} = \vec{\mathbf{0}}$$

Result: $(2x-4) = 0 \quad (3y-9) = 0 \quad (\text{yellow box}) = 0$ Hint: think $\times \cdot$,
 $x = 2 \quad y = 3 \quad z = 4$ not matrix algebra.

1.30 ♣ **Number of independent scalar equations from 1 vector equation.** (Section 2.9.5)

The **vector** equation shown right is useful for static analyses of a system S .

$$\vec{\mathbf{F}}^S = \vec{\mathbf{0}}$$

In the table to the right, box all integers that could be equal to the number of **independent scalar** equations produced by the previous vector equation. Hint: Hw 1.29. Related Hw 13.13.

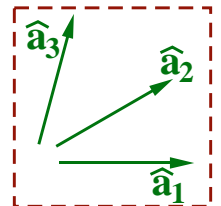
System type	Integer(s)
1D (line)	0 1 2 3 4 ⁺
2D (planar)	0 1 2 3 4 ⁺
3D (spatial)	0 1 2 3 4 ⁺

Note: 1D/linear means $\vec{\mathbf{F}}^S$ can be expressed in terms of one vector $\hat{\mathbf{i}}$.
 2D/planar means $\vec{\mathbf{F}}^S$ can be expressed in terms of two non-parallel unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.
 3D/spatial means $\vec{\mathbf{F}}^S$ can be expressed in terms of three non-coplanar unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

1.31 ♣ **Vector concepts: Solving a vector equation** (just circle true or false and fill-in the blank).

Consider the following vector equation written in terms of the scalars x, y, z and three unique non-orthogonal **coplanar** unit vectors $\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3$.

$$(2x-4)\hat{\mathbf{a}}_1 + (3y-9)\hat{\mathbf{a}}_2 + (4z-16)\hat{\mathbf{a}}_3 = \vec{\mathbf{0}}$$



The **unique** solution to this vector equation is $x = 2, y = 3, z = 4$. **True/False.**

Explain: $\hat{\mathbf{a}}_2$ can be expressed in terms of $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$ (i.e., $\hat{\mathbf{a}}_2$ is a linear combination of $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$). Hence the vector equation produces linearly independent scalar equations.

1.32 ♣ **Concept: A vector revolution for geometry.** (Chapter 2)

The relatively new invention of vectors (Gibbs \approx 1900 AD) has revolutionized Euclidean geometry (Euclid \approx 300 BC). For each geometrical quantity below, circle the vector operation(s) (either the dot-product, cross-product, or both) that is **most** useful for their calculation.

Length: • ×	Angle: • ×
Area: • ×	Volume: <input checked="" type="checkbox"/> <input checked="" type="checkbox"/>

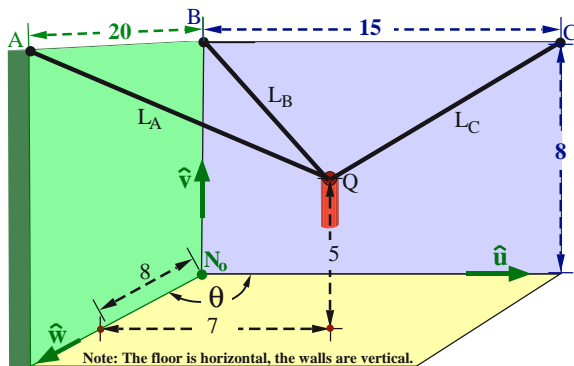
1.33 † **Microphone cable lengths (non-orthogonal walls) “It’s just geometry”.** Show work.

A microphone Q is attached to three pegs A, B, C by three cables. Knowing the peg locations, microphone location, and the angle θ between the vertical walls, express L_A, L_B, L_C solely in terms of numbers and θ . Next, complete the table by calculating L_B when $\theta = 120^\circ$.

Hint: To do this **efficiently**, use only unit vectors $\hat{u}, \hat{v}, \hat{w}$, and (do **not** introduce an **orthogonal** set of unit vectors).

Hint: Use the distributive property of the vector dot-product as shown in Section 2.9.1 and Homework 2.4.

Note: Synthesis problems are difficult. Think, talk, draw, sleep, walk, get help, ... (if necessary, read Section 3.2).



Distance between A and B	20 m
Distance between B and C	15 m
Distance between N_o and B	8 m
Distance along back wall (see picture)	7 m
Q 's height above N_o	5 m
Distance along side wall (see picture)	8 m
L_A : Length of cable joining A and Q	16.9 m
L_B : Length of cable joining B and Q	8.1 m
L_C : Length of cable joining C and Q	14.2 m

$${}^{N_o}\vec{r}^Q = 7\hat{u} + 5\hat{v} + 8\hat{w}$$

Result: $L_A = \sqrt{202 - 168 \cos(\theta)}$ $L_B =$ $L_C = \sqrt{137 -$ $\cos(\theta)$

- Knowing β is defined as the angle between lines $\overline{BN_o}$ and \overline{BQ} , show $\beta \approx$.

Vector addition, dot products, and cross products: + · ×

Show work – except for ♣ fill-in-blanks (print .pdf from www.MotionGenesis.com ⇒ [Textbooks](#) ⇒ [Resources](#)).

2.1 ♣ Right-handed, orthogonal, unit vectors. (Section 4.1)

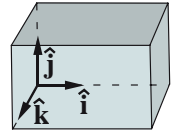
Draw a set of right-handed orthogonal (mutually perpendicular) unit vectors consisting of $\hat{n}_x, \hat{n}_y, \hat{n}_z$. In other words, draw $\hat{n}_x, \hat{n}_y, \hat{n}_z$ so that \hat{n}_y is perpendicular (orthogonal) to \hat{n}_x and $\hat{n}_z = \hat{n}_x \times \hat{n}_y$.



2.2 ♣ Adding and subtracting vectors. (Sections 2.6, 2.8)

Shown right are unit vectors $\hat{i}, \hat{j}, \hat{k}$ and vectors $\vec{p}, \vec{q}, \vec{r}$. Form the vector sums and differences below.

$$\begin{aligned} \vec{p} &= a\hat{i} + b\hat{j} + c\hat{k} \\ \vec{q} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \vec{r} &= 5\hat{i} - 6\hat{j} + 7\hat{k} \end{aligned}$$



Result:

$$\begin{aligned} \vec{p} + \vec{q} &= (a+x)\hat{i} + \boxed{}\hat{j} + \boxed{}\hat{k} & \vec{p} - \vec{q} &= (a-x)\hat{i} + \boxed{}\hat{j} + \boxed{}\hat{k} \\ \vec{q} + \vec{r} &= \boxed{}\hat{i} + \boxed{}\hat{j} + \boxed{}\hat{k} & \vec{q} - \vec{r} &= \boxed{}\hat{i} + \boxed{}\hat{j} + \boxed{}\hat{k} \end{aligned}$$

2.3 ♣ Words: Vectors and column matrices (Section 2.1, Hw 1.2)

As defined by Gibbs, physical vectors have magnitude and direction. True/False

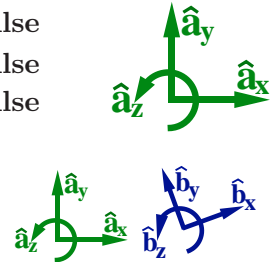
In math (linear algebra), a column matrix is a vector. True/False

The vector $\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z$ is equal to the column matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. True/False

($\hat{a}_x, \hat{a}_y, \hat{a}_z$ are the orthogonal unit vectors shown right).

$$\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z + 4\hat{b}_x + 5\hat{b}_y + 6\hat{b}_z = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \quad \text{True/False}$$

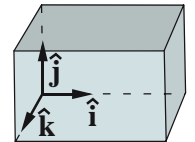
($\hat{a}_x, \hat{a}_y, \hat{a}_z$ and $\hat{b}_x, \hat{b}_y, \hat{b}_z$ are shown right).



2.4 ♣ Dot products with orthogonal unit vectors. (Sections 2.9, 2.9.3)

Given: Right-handed orthogonal unit vectors $\hat{i}, \hat{j}, \hat{k}$ and:

$$\begin{aligned} \vec{v} &= a\hat{i} + b\hat{j} + c\hat{k} \\ \vec{w} &= x\hat{i} + y\hat{j} + z\hat{k} \end{aligned}$$



- Use the *distributive property* for dot products to write $\vec{v} \cdot \vec{w}$ in terms of $\hat{i} \cdot \hat{i}, \hat{i} \cdot \hat{j}$, etc. Next, use the *definition* of the dot product to calculate $\hat{i} \cdot \hat{i}, \hat{i} \cdot \hat{j}$, etc. (below-right).

Result: $\vec{v} \cdot \vec{w} = (a\hat{i} + b\hat{j} + c\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$

$$\begin{aligned} \vec{v} \cdot \vec{w} &= ax\hat{i} \cdot \hat{i} + ay\hat{i} \cdot \hat{j} + \boxed{}\hat{i} \cdot \hat{k} \\ &+ bx\hat{j} \cdot \hat{i} + by\boxed{} \cdot \boxed{} + \boxed{}\boxed{} \cdot \boxed{} \\ &+ cx\hat{k} \cdot \hat{i} + cy\boxed{} \cdot \boxed{} + \boxed{}\boxed{} \cdot \boxed{} \end{aligned}$$

$\hat{i} \cdot \hat{i} = 1$	$\hat{i} \cdot \hat{j} = \boxed{}$	$\hat{i} \cdot \hat{k} = \boxed{}$
$\hat{j} \cdot \hat{i} = 0$	$\hat{j} \cdot \hat{j} = \boxed{}$	$\hat{j} \cdot \hat{k} = \boxed{}$
$\hat{k} \cdot \hat{i} = 0$	$\hat{k} \cdot \hat{j} = \boxed{}$	$\hat{k} \cdot \hat{k} = \boxed{}$

- Simplify $\vec{v} \cdot \vec{w}$ and use its special dot-product formula for the calculations that follow.

Result:

$$\vec{v} \cdot \vec{w} = ax + by + \boxed{}$$

This special dot-product formula is an efficient way to calculate $\vec{v} \cdot \vec{w}$ when $\hat{i}, \hat{j}, \hat{k}$ are **orthogonal unit** vectors.

Given

$$\begin{aligned} \vec{p} &= 2\hat{i} + 3\hat{j} + 4\hat{k} \\ \vec{q} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \vec{r} &= 5\hat{i} - 6\hat{j} + 7\hat{k} \end{aligned}$$

Calculate

$$\begin{aligned} \vec{p} \cdot \vec{q} &= 2x + 3y + \boxed{}z \\ \vec{p} \cdot \vec{r} &= \boxed{} \\ \vec{q} \cdot \vec{r} &= \boxed{} \end{aligned}$$

$$\begin{aligned} |\vec{p}| &= \sqrt{29} \\ |\vec{q}| &= \sqrt{\boxed{}} \\ |\vec{r}| &= \sqrt{110} \end{aligned}$$

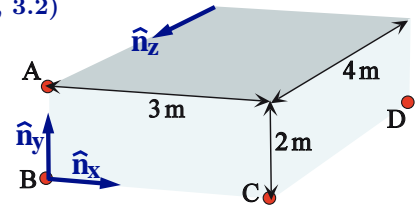
2.5 ♣ **Perpendicular vectors.** (Note: $\hat{i}, \hat{j}, \hat{k}$ are orthogonal unit vectors). (Section 2.9)

Draw two vectors \vec{v} and \vec{w} that are perpendicular. Hence, $\vec{v} \cdot \vec{w} = \square$.
 When $\vec{v} = x\hat{i} + 2\hat{j} + 3\hat{k}$ is perpendicular to $\vec{w} = 4\hat{i} + 5\hat{j} + 6\hat{k}$, $x = \square$.



2.6 **Dot products to calculate distance and angles.** (Sections 2.9, 3.2)

The figure to the right shows a block with sides of length 2 m, 3 m, 4 m and points A, B, C located at corners. Right-handed orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$ are directed with \hat{n}_x from B to C and \hat{n}_y from B to A .



- (a) Express \vec{r} (position from A to C) in terms of $\hat{n}_x, \hat{n}_y, \hat{n}_z$ and find a numerical value for $|\vec{r}|^2$.
 Next calculate the distance d between A to C (magnitude of \vec{r}).

Result: $\vec{r} = \square \hat{n}_x - \square \hat{n}_y$ $|\vec{r}|^2 = \vec{r} \cdot \vec{r} = \square \text{ m}^2$ $d = \sqrt{\square} \text{ m}$

- (b) Calculate the unit vector \hat{u} directed from A to C and the unit vector \hat{v} directed from A to D .

Result: $\hat{u} = \frac{3\hat{n}_x - \square \hat{n}_y}{\sqrt{\square}}$ $\hat{v} = \frac{\square \hat{n}_x - \square \hat{n}_y - \square \hat{n}_z}{\sqrt{\square}}$

- (c) Calculate $\angle BAC$, the angle between line \overline{AB} and line \overline{AC} .
 Next, calculate $\angle CAD$, the angle between line \overline{AC} and line \overline{AD} .

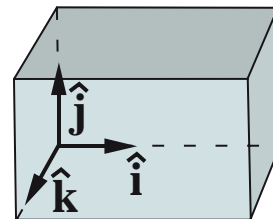
Result: $\angle BAC = \square^\circ$ $\angle CAD = \square^\circ$

2.7 ♣ **Construct a unit vector \hat{u} in the direction of each vector given below.** (Section 2.9.2)

Vector	Unit vector \hat{u}
$3\hat{i}$	\hat{i}
$-3\hat{i}$	\square
$3\hat{i} - 4\hat{j}$	$\frac{\square \hat{i} - \square \hat{j}}{\square}$
$3\hat{i} - 4\hat{j} + 12\hat{k}$	$\frac{\square \hat{i} - \square \hat{j} + \square \hat{k}}{\square}$
$c\hat{i}$	$\frac{c}{ c } \hat{i}$ or $\text{sign}(c)\hat{i}$

c is a real non-zero number

Note: $\hat{i}, \hat{j}, \hat{k}$ are orthogonal unit vectors.



Ensure your last answer agrees with your first two answers, e.g., if $c = 3$ or $c = -3$.

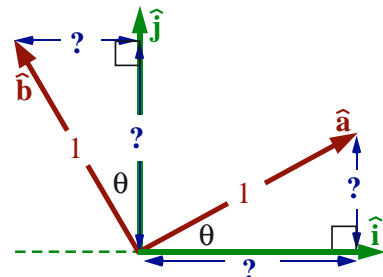
2.8 ♣ **Vector components: Sine and cosine.** (Section 1.4)

Replace each $?$ in the figure to the right with $\sin(\theta)$ or $\cos(\theta)$.
 Next, use vector addition to express unit vectors \hat{a} and \hat{b} in terms of $\sin(\theta), \cos(\theta), \hat{i}, \hat{j}$.

Result: $\hat{a} = \square \hat{i} + \square \hat{j}$

Reminder: $\hat{b} = \square \hat{i} + \square \hat{j}$

SohCahToa

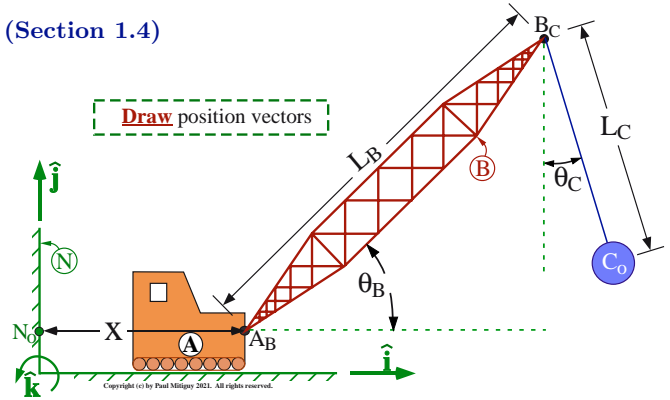


2.9 ♣ Vector components for a crane-boom. (Section 1.4)

Shown right is a crane whose cab A supports a boom B that swings a wrecking ball C_o .

Right-handed orthogonal unit vectors $\hat{i}, \hat{j}, \hat{k}$ are directed with \hat{i} horizontally-right, \hat{j} vertically-upward, and \hat{k} outward-normal to the plane containing points N_o, A_B, B_C, C_o .

Draw each position vector listed below and then use your knowledge of sine/cosine to resolve these vectors into \hat{i} and \hat{j} components.



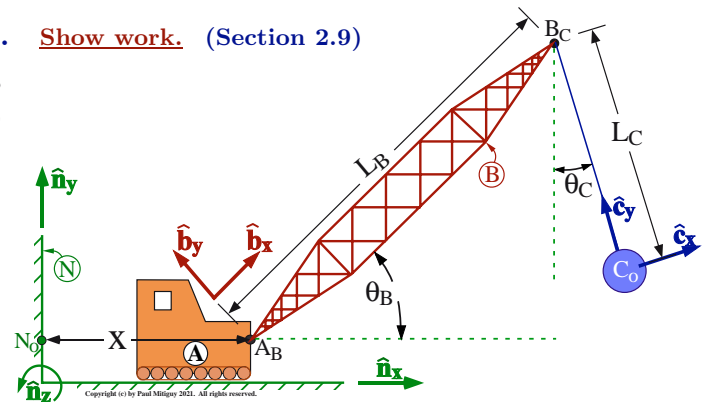
- Position from N_o to A_B ${}^{N_o}\vec{r}^{A_B} =$ $\square \hat{i} +$ $\square \hat{j}$
- Position from A_B to B_C ${}^{A_B}\vec{r}^{B_C} =$ $\square \hat{i} +$ $\square \hat{j}$
- Position from B_C to C_o ${}^{B_C}\vec{r}^{C_o} =$ $\square \hat{i} +$ $\square \hat{j}$
- Position from N_o to B_C ${}^{N_o}\vec{r}^{B_C} =$ $\square \hat{i} +$ $\square \hat{j}$
- Position from N_o to C_o ${}^{N_o}\vec{r}^{C_o} =$ $\square \hat{i} + [L_B \sin(\theta_B) - L_C \cos(\theta_C)] \hat{j}$

2.10 Dot products and distance calculations. **Show work.** (Section 2.9)

Shown right is a crane whose cab A supports a boom B that swings a wrecking ball C_o . To prevent the wrecking ball from hitting a car, the distance between N_o and point B_C (the tip of the boom) must be controlled.

To start this problem, express \vec{r} (the position vector from N_o to B_C) in terms of $x, L_B, \hat{n}_x, \hat{b}_x$.

Result: $\vec{r} = \square \hat{n}_x + \square \hat{b}_x$



• **Without** resolving \vec{r} into \hat{n}_x and \hat{n}_y components, use $|\vec{r}| = \sqrt{\vec{r} \cdot \vec{r}}$ [from equation (3.1)] and the distributive property to calculate the distance between N_o and B_C in terms of x, L_B, θ_B .

Result: (if stumped, hint below).¹ **Optional:** Calculate $|\vec{r}|$ when $x = 20$ m, $L_B = 10$ m, $\theta_B = 30^\circ$.

Distance between N_o and B_C : $|\vec{r}| = \sqrt{\square^2 + \square^2 + 2xL_B \cos(\theta_B)} \approx 29.1$ m

• Homework 2.9 showed \vec{r} can be expressed as $\vec{r} = [x + L_B \cos(\theta_B)] \hat{n}_x + L_B \sin(\theta_B) \hat{n}_y$. Use this expression to verify your previous result for $|\vec{r}| = \sqrt{\vec{r} \cdot \vec{r}}$.

Result: $|\vec{r}|$ simplifies to the previous result but uses an inefficient process and $\sin^2(\theta_B) + \cos^2(\theta_B) = 1$.

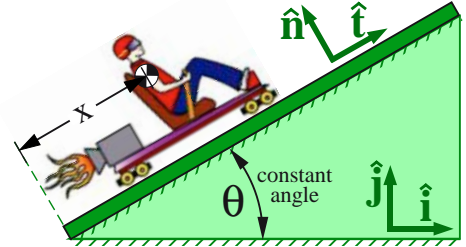
¹Hint: The distributive property for vector dot-multiplication is $(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$. Use the distributive property to express $\vec{r} \cdot \vec{r}$ in terms of x, L_B , and $\hat{n}_x \cdot \hat{b}_x$. Thereafter, use the **dot-product definition** of $(\hat{n}_x \cdot \hat{b}_x)$ to form $\vec{r} \cdot \vec{r} = \square^2 + \square^2 + 2xL_B(\hat{n}_x \cdot \hat{b}_x) \stackrel{(2.2)}{=} \square^2 + \square^2 + 2xL_B \cos(\square)$.

2.11 Vector components, free-body diagram (FBD), and motion graphs for a rocket-sled.

The following figure shows a rocket-sled moving along smooth (frictionless) inclined rails.

\hat{t} is a unit vector tangent to the rails and \hat{n} is normal to the rails.

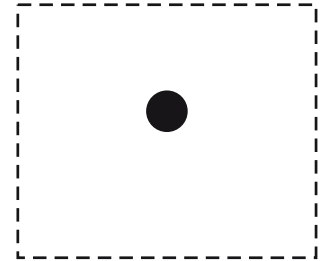
Description	Symbol	Type
Mass of rocket-sled and rider	m	Constant
Earth's gravitational acceleration	g	Constant
Angle between horizontal and inclined-rails	θ	Constant
\hat{t} measure of rocket-sled position	x	Variable
\hat{n} measure of normal force on sled	F_n	Variable
\hat{t} measure of thrust force on sled	F_T	Specified



FBD Draw a free-body diagram with a particle representing the rocket sled. Draw the contact and distance forces on the particle.

Result:

$$\vec{F}_{\text{Net}} = \underbrace{F_T \hat{t}}_{\vec{F}_{\text{Thrust}}} + \underbrace{F_n \hat{n}}_{\vec{F}_{\text{Normal}}} - \underbrace{mg \hat{j}}_{\vec{F}_{\text{gravity}}}$$



$\vec{r}, \vec{v}, \vec{a}$ Form the rocket-sled's position, velocity, and acceleration.

Result:

$$\vec{r} = x \hat{t} \quad \vec{v} = \dot{x} \hat{t} \quad \vec{a} = \ddot{x} \hat{t}$$

Substitute \vec{F}_{Net} and \vec{a} into $\vec{F}_{\text{Net}} = m \vec{a}$ and form scalar equations.

Result:

$$\underbrace{[F_T - mg \sin(\theta)] \hat{t}}_{\vec{F}_{\text{Net}}} + \underbrace{[F_n - mg \cos(\theta)] \hat{n}}_{\vec{F}_{\text{Net}}} = \underbrace{m \ddot{x} \hat{t}}_{m \vec{a}}$$

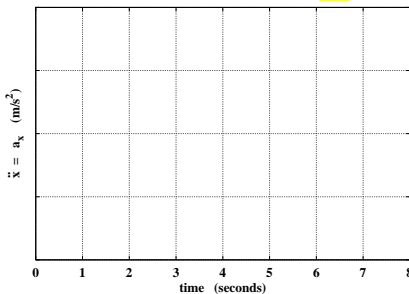
\hat{t} direction: $[F_T - mg \sin(\theta)] = m \ddot{x}$ $\ddot{x} = [F_T - mg \sin(\theta)] / m$

\hat{n} direction: $F_n - mg \cos(\theta) = 0$ \Rightarrow Solve $F_n = mg \cos(\theta)$

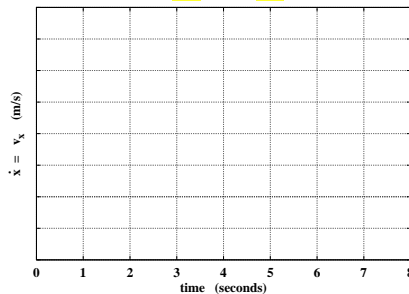
Solve Knowing $m = 100 \text{ kg}$, $g = 10 \frac{\text{m}}{\text{s}^2}$, $\theta = 30^\circ$, $F_T = 700 \text{ N}$, determine \ddot{x} and graph it.

Knowing the rocket-sled starts at $x = 8 \text{ m}$ and is initially moving backward (downward-left) along the rail at $4 \frac{\text{m}}{\text{s}}$, determine $\dot{x}(t)$ and $x(t)$ and **sketch** them for $0 \leq t \leq 8$.

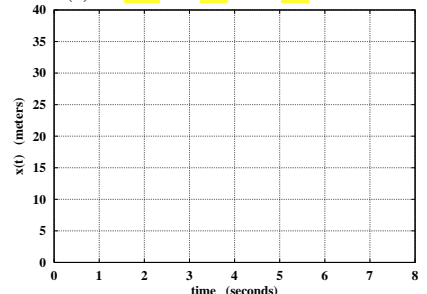
Result: $\ddot{x}(t) = 1 \text{ m/s}^2$



$\dot{x}(t) = 4 \text{ m/s} - 1 \text{ m/s}^2 t$



$x(t) = 8 \text{ m} - 4 \text{ m/s} t + \frac{1}{2} \text{ m/s}^2 t^2$



Include friction between the rails and rocket-sled, modeled via a coefficient of kinetic friction μ_k . Express \ddot{x} and F_n in terms of some/all of μ_k and symbols in the table. Knowing $\mu_k \approx 0.115$, $x(0) = 8 \text{ m}$, and the rocket-sled initially moves **upward-right** at $4 \frac{\text{m}}{\text{s}}$, find $\dot{x}(t)$ and $x(t)$.

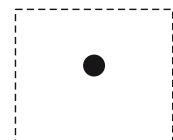
Result:

$$\ddot{x}(t) = \frac{F_T - mg \sin(\theta) - \mu_k mg \cos(\theta)}{m} \approx 1 \frac{\text{m}}{\text{s}^2} \quad F_n = mg \cos(\theta) \quad \dot{x}(t) \approx 4 \text{ m/s} - 1 \text{ m/s}^2 t$$

$$x(t) \approx 8 \text{ m} - 4 \text{ m/s} t + \frac{1}{2} \text{ m/s}^2 t^2$$

Calculate the minimum thrust (redraw FBDs) to:

	Result
a. Keep the rocket-sled moving uphill at constant speed	$F_T \approx 100 \text{ N}$
b. Keep the rocket-sled moving downhill at constant speed	$F_T \approx 100 \text{ N}$



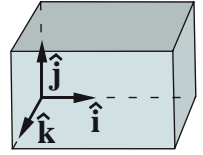
For part b, assume the rocket is initially moving **downward-left** at 4 m/s .
Solution at www.MotionGenesis.com \Rightarrow [Get Started](#) \Rightarrow [Rocket sled](#).

2.12 ♣ **Cross products with right-handed orthogonal unit vectors.** (Section 2.10)

Given: Right-handed orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$ and:

$$\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\vec{w} = x\hat{i} + y\hat{j} + z\hat{k}$$



• Use the **distributive property** for cross products to write $\vec{v} \times \vec{w}$ in terms of $\hat{i} \times \hat{i}, \hat{i} \times \hat{j}$, etc.

Result:

$$\vec{v} \times \vec{w} = ax\hat{i} \times \hat{i} + ay\hat{i} \times \hat{j} + az\hat{i} \times \hat{k}$$

$$+ \text{[yellow box]} + \text{[yellow box]} + \text{[yellow box]}$$

$$+ \text{[yellow box]} + \text{[yellow box]} + \text{[yellow box]}$$

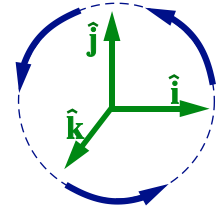
• Use the **definition** of the cross product to calculate $\hat{i} \times \hat{i}, \hat{i} \times \hat{j}$, etc.

Result:

$$\hat{i} \times \hat{i} = \vec{0} \quad \hat{i} \times \hat{j} = \hat{k} \quad \hat{i} \times \hat{k} = -\hat{j}$$

$$\hat{j} \times \hat{i} = \text{[yellow box]} \quad \hat{j} \times \hat{j} = \text{[yellow box]} \quad \hat{j} \times \hat{k} = \text{[yellow box]}$$

$$\hat{k} \times \hat{i} = \text{[yellow box]} \quad \hat{k} \times \hat{j} = \text{[yellow box]} \quad \hat{k} \times \hat{k} = \text{[yellow box]}$$



• In view of your previous two results, calculate $\vec{v} \times \vec{w}$.

Result:

$$\vec{v} \times \vec{w} = (\text{[yellow box]})\hat{i} + (\text{[yellow box]})\hat{j} + (\text{[yellow box]})\hat{k}$$

2.13 **Cross products and determinants.** (Section 2.10.2)

Shown right, arbitrary vectors \vec{a} and \vec{b} are expressed in terms of right-handed orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$.

$$\vec{a} = a_x\hat{n}_x + a_y\hat{n}_y + a_z\hat{n}_z$$

$$\vec{b} = b_x\hat{n}_x + b_y\hat{n}_y + b_z\hat{n}_z$$

Show that calculating $\vec{a} \times \vec{b}$ with the **distributive property** of the cross product (seen in Hw 2.12) happens to be equal to the **determinant** of the matrix shown to the right.

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$

Next, use this determinant method to calculate $\vec{v} \times \vec{w}$, etc.

Result:

$$\vec{u} \times \vec{v} = (3z - 4y)\hat{n}_x + (4x - 2z)\hat{n}_y + (\text{[yellow box]})\hat{n}_z$$

$$\vec{v} \times \vec{w} = (\text{[yellow box]})\hat{n}_x + (\text{[yellow box]})\hat{n}_y - (\text{[yellow box]})\hat{n}_z$$

Optional: $\vec{u} \times \vec{w} = 45\hat{n}_x + 6\hat{n}_y - 27\hat{n}_z$

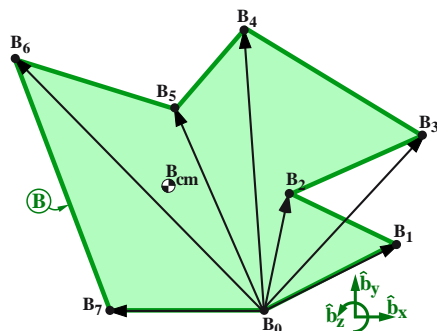
2.14 ♣ **Cross products: Commercial algorithm for area calculations (surveying).** (Section 2.10.1)

Complex **planar objects** such as the polygon B below can be decomposed into triangles for important planar measurements (e.g., farming acreage, building costs, and mass and area properties of 2D objects).



- Calculate \vec{A}_2 and \vec{A}_4 , the vector-areas of triangles $B_0 B_2 B_3$ and $B_0 B_4 B_5$.
- Account for overlapped areas with **positive** and **negative** vector areas.

Result: (Just fill in the calculations for \vec{A}_2, \vec{A}_4 , and \vec{A}).



$$\vec{r}_1 = B_0\vec{r}^{B_1} = 2.0\hat{b}_x + 2.0\hat{b}_y$$

$$\vec{r}_2 = B_0\vec{r}^{B_2} = 0.5\hat{b}_x + 2.5\hat{b}_y$$

$$\vec{r}_3 = B_0\vec{r}^{B_3} = 3.0\hat{b}_x + 4.0\hat{b}_y$$

$$\vec{r}_4 = B_0\vec{r}^{B_4} = -0.5\hat{b}_x + 7.0\hat{b}_y$$

$$\vec{r}_5 = B_0\vec{r}^{B_5} = -1.0\hat{b}_x + 5.0\hat{b}_y$$

$$\vec{r}_6 = B_0\vec{r}^{B_6} = -3.0\hat{b}_x + 6.0\hat{b}_y$$

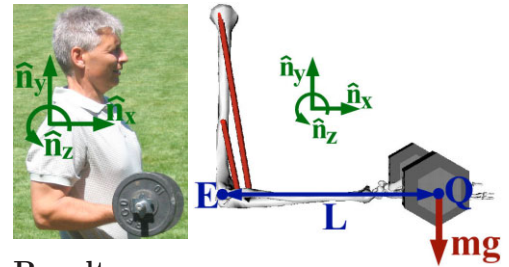
$$\vec{r}_8 = B_0\vec{r}^{B_8} = -2.0\hat{b}_x + 0.0\hat{b}_y$$

$\vec{A}_1 = \frac{1}{2} \vec{r}_1 \times \vec{r}_2 = 2\hat{b}_z$	
$\vec{A}_2 = \frac{1}{2} \vec{r}_2 \times \vec{r}_3 =$ [yellow box]	
$\vec{A}_3 = \dots = 11.5\hat{b}_z$	
$\vec{A}_4 = \dots =$ [yellow box]	
$\vec{A}_5 = \dots = 4.5\hat{b}_z$	
$\vec{A}_6 = \frac{1}{2} \vec{r}_6 \times \vec{r}_7 = 6\hat{b}_z$	
$\vec{A} = \sum_{i=1}^6 \vec{A}_i =$ [yellow box]	
Area = $ \vec{A} = 23.5$	

2.15 Biomechanics: Gravity moment for curling $\vec{M} = \vec{r} \times \vec{F}$ (Section 2.10)

The figures to the right show an athlete curling a dumbbell (modeled as a particle Q of mass m). The forearm connects to the upper arm at the elbow (point E). Right-handed unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$ are directed with \hat{n}_y vertically upward and \hat{n}_x from E to Q .

Description	Symbol	Type
Earth's gravitational constant	g	$g \approx 9.8 \frac{m}{s^2}$
Mass of dumbbell Q	m	Positive constant
Distance between elbow E and Q	L	Positive constant



Result:

$\vec{M} = \vec{r} \times \vec{F} =$

The moment of gravity forces on Q about E is $\vec{M} = \vec{r} \times \vec{F}$ where $\vec{F} = -mg \hat{n}_y$. Express \vec{M} in terms of m, g, L .

Now consider the forearm making an angle θ with downward vertical. Unit vector \hat{b}_y is directed from Q to E and $\hat{b}_z = \hat{n}_z$.

Form $\vec{M} = \vec{r} \times \vec{F}$, its magnitude $|\vec{M}|$, and the values of θ ($0 \leq \theta \leq 135^\circ$) that produce maximum and minimum $|\vec{M}|$.

To simplify $|\vec{M}|$, note m, g, L are positive and for $0 \leq \theta \leq 135^\circ$, $\sin(\theta) \geq 0$.

Result: (in terms of m, g, L, θ).

$\vec{M} = \vec{r} \times \vec{F} =$ $|\vec{M}| =$



Max $|\vec{M}| =$ at $\theta =$ $^\circ$
 Min $|\vec{M}| =$ at $\theta =$ $^\circ$

Optional: Modeling the elbow as a revolute joint, draw a *free-body diagram (FBD)* of the system consisting of the forearm and dumbbell.

2.16 Biomechanics: Gravity force and moment for tennis $\vec{M} = \vec{r} \times \vec{F}$ (Section 2.10)

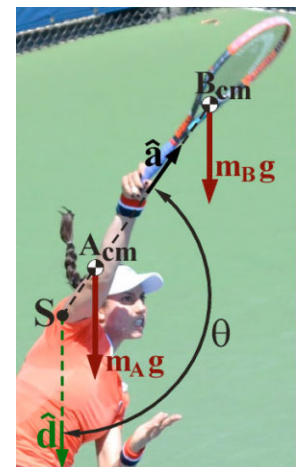
Shown right is an athlete whose arm A swings a tennis racquet B . Point S (shoulder), A_{cm} (A 's center of mass), and B_{cm} (B 's center of mass) lie along a line parallel to a unit vector \hat{a} . The unit vector \hat{d} is vertically-downward \downarrow .

Description	Symbol	Type
Earth's gravitational constant	g	$g \approx 9.8 \frac{m}{s^2}$
Mass of A , mass of B	m_A, m_B	Positive constants
Distances between S and A_{cm} and S and B_{cm}	L_A, L_B	Positive constants
Angle between \hat{a} and \hat{d}	θ	$0 \leq \theta \leq 180^\circ$

- Form $\vec{F}_{gravity}$ (the net force on A and B due to Earth's gravity).
- Form $|\vec{M}|$ (the magnitude of the moment of those gravity forces about S).

Note: $\vec{M} = {}^S\vec{r}^{A_{cm}} \times m_A g \hat{d} + {}^S\vec{r}^{B_{cm}} \times m_B g \hat{d}$.

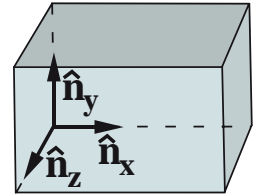
Result: $\vec{F}_{gravity} =$ \hat{d}
 $|\vec{M}| =$



Optional: Modeling the athlete grip of the racquet as a weld, draw a *free-body diagram (FBD)* of the racquet. Next, choose a model for the shoulder joint and draw a *FBD* of the system consisting of the arm and racquet.

2.17 Scalar triple product with bases (Section 2.11).

The figure shows right-handed orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$.



Given

$$\vec{u} = 2\hat{n}_x + 3\hat{n}_y + 4\hat{n}_z$$

$$\vec{v} = x\hat{n}_x + y\hat{n}_y + z\hat{n}_z$$

$$\vec{w} = 5\hat{n}_x - 6\hat{n}_y + 7\hat{n}_z$$

Calculate

$$\vec{u} \times \vec{v} \cdot \vec{u} = \text{[]}$$

$$\vec{u} \times \vec{v} \cdot \vec{w} = \text{[]}z - \text{[]}x - 6y$$

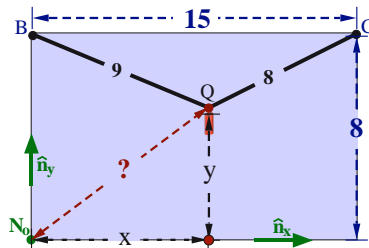
$$\vec{u} \cdot \vec{v} \times \vec{w} = \text{[]}z - 45x - \text{[]}y$$

Note: Although the order of operations in $\vec{u} \times \vec{v} \cdot \vec{u}$ is unambiguous, parentheses may clarify your work.

$\vec{u} \times \vec{v} \cdot \vec{w} = \vec{u} \cdot \vec{v} \times \vec{w}$ and it is OK to switch \cdot and \times in scalar triple products. **True/False**.

2.18 Locating a microphone (2D). Show work. (Section 1.4)

A microphone Q is attached to two pegs B and C by two cables. Knowing the peg locations, cable lengths, and points B, C, Q, N_o all lie in the same plane, determine the distance between Q and N_o . Do the problem with Euclidean geometry (e.g., law of cosines), then try vectors (see Hw 1.33).

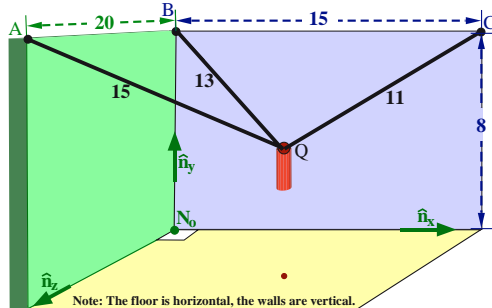


Distance between B to C		15 m
Distance between N_o to B	h	8 m
Length of cable joining B and Q	L_B	9 m
Length of cable joining C and Q	L_C	8 m
Distance between N_o and Q		9.01 m

Note: Although there are two mathematical answers to this problem, one is above the ceiling by ≈ 12 m and requires the cables to be in compression.

2.19 † Locating a microphone (3D).

A microphone Q is attached to three pegs $A, B,$ and C by three cables. Knowing the peg locations and cable lengths, determine the distance between Q and point N_o . **Show work.** (If needed, hint below).²



Distance between A to B		20 m
Distance between B to C		15 m
Distance between N_o to B	h	8 m
Length of cable joining A and Q	L_A	15 m
Length of cable joining B and Q	L_B	13 m
Length of cable joining C and Q	L_C	11 m
Distance between N_o and Q		13.3 m
If Q is above ceiling, distance ≈ 17 m		

Note: This is part of the process of a camera targeting a football/baseball in a stadium or laser targeting cancer or ...

2.20 Optional: Draw the free-body diagram (FBD) for each object below.

Particle Q Hw 2.19	Top block Hw 12.12	Bottom pulley Hw 12.14	Rolling spool B Hw 13.12	Bureau B Hw 13.8	Entire system Hw 11.17

²Hint: See Homework 1.33 or Section 3.2. Introduce unknowns x, y, z so Q 's position from N_o is $x\hat{n}_x + y\hat{n}_y + z\hat{n}_z$. Although nonlinear algebraic equations are usually solved with a computer, these can also be solved "by-hand".

Solution at www.MotionGenesis.com \Rightarrow [Get Started](#) \Rightarrow 2D/3D geometry. Alternatively, go to www.WolframAlpha.com and type Solve $x^2 + (-20+z)^2 + (-8+y)^2 = 225, \quad x^2 + z^2 + (-8+y)^2 = 169, \quad z^2 + (-15+x)^2 + (-8+y)^2 = 121$