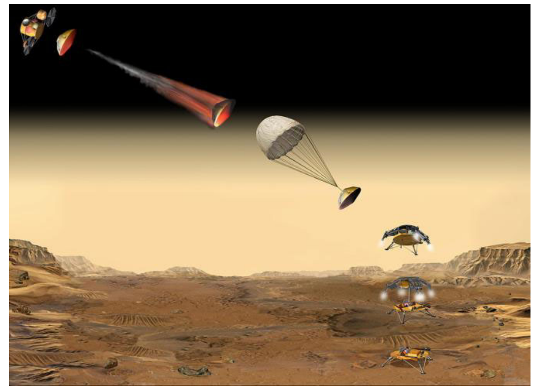


Courtesy NASA/JPL-Caltech

Chapter 2

Vectors

Summary (see examples in Hw 1, 2, 3)

In 1881-1903, Gibbs developed ***vectors*** as a useful combination of magnitude and direction. Vectors are an important **geometrical tool** [useful for surveying, motion analysis, lasers, optics, computer graphics, animation, CAD/CAE (computer aided drawing/engineering), FEA(finite element analysis), . . .].

Symbol	Description	Details
$\vec{\mathbf{0}}$, $\widehat{\mathbf{u}}$	Zero vector and unit vector.	Sections 2.3, 2.4
$+ \; - \; *$	Vector addition, negation, subtraction, and multiplication/division with a scalar.	Sections 2.6 - 2.8
$\cdot \quad \times$	Vector dot product and cross product.	Sections 2.9, 2.10
$\frac{{}^F d}{dt}$	Vector differentiation.	Chapters 6, 7

2.1 Examples of scalars, vectors, and dyadics

- A ***scalar*** is a non-directional quantity (e.g., a real number). Examples include:

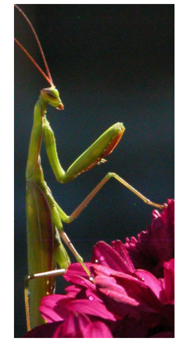
time	density	volume	mass	potential energy	work
distance	speed	angle	weight	kinetic energy	temperature

- A ***vector*** is a quantity that has magnitude and ***one*** associated direction. For example, a ***velocity vector*** has both speed (how fast something moves) and direction (which way it is going). A ***force vector*** has magnitude (how hard something is pushed) and direction (which way it is shoved). Examples include:

force	velocity	acceleration	linear momentum
torque	angular velocity	angular acceleration	angular momentum

- A ***dyad*** is a quantity with magnitude and ***two*** associated directions. For example, ***stress*** associates with area and force (both regarded as vectors). A ***dyadic*** is the **sum of dyads**. For example, an ***inertia dyadic*** (Chapter 14) is the sum of dyads associated with moments and products of inertia.

2.2 Definition of a vector

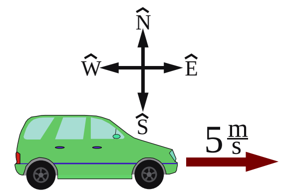


Courtesy Bro. Claude Rheaume LaSalette.

A ***vector*** is defined as a quantity having ***magnitude*** and ***direction***.[a]
Vectors are represented pictorially with straight or curved arrows (examples below).
Vectors are typeset with an arrow and bold-faced font, e.g., $\vec{\mathbf{v}}$ denotes a vector.

Certain vectors have additional special properties. For example, a ***position vector*** $\vec{\mathbf{r}}$ is associated with two points and has units of length (e.g., meters).

[a] A vector's ***magnitude*** is a real non-negative number. A vector's ***direction*** can be resolved into ***orientation*** and ***sense***. For example, a highway has an orientation (e.g., east-west) and a vehicle traveling east has a sense. Knowing both the orientation of a line and the sense on the line gives direction. Changing a vector's orientation or sense changes its direction.



Example of a vector: Consider the statement "the car is moving East at 5 $\frac{\mathrm{m}}{\mathrm{s}}$". It is convenient to represent the car's speed and direction with the velocity vector $\vec{\mathbf{v}} = 5\,\widehat{\mathbf{East}}$ (a hat designates the direction $\widehat{\mathbf{East}}$ as a ***unit vector***). The car's speed is always a real non-negative number equal to $|\vec{\mathbf{v}}|$ (the ***magnitude*** of $\vec{\mathbf{v}}$). The combination of ***magnitude*** and ***direction*** is a ***vector***.

The velocity of a car with speed 5 $\frac{\mathrm{m}}{\mathrm{s}}$ moving West can also be written as $\vec{\mathbf{v}} = -5\,\widehat{\mathbf{East}}$. The negative sign in $-5\,\widehat{\mathbf{East}}$ is associated with the vector's direction whereas the magnitude of $\vec{\mathbf{v}}$ is $|\vec{\mathbf{v}}| = |-5\,\widehat{\mathbf{East}}| = 5\,\frac{\mathrm{m}}{\mathrm{s}}$.
Note: When $\vec{\mathbf{v}}$ is written as $\vec{\mathbf{v}} = \dot{x}\,\widehat{\mathbf{East}}$ where $\dot{x}$ is a scalar that can be **positive** or **zero** or **negative**, $\dot{x}$ is called the $\widehat{\mathbf{East}}$ ***measure*** of the vector $\vec{\mathbf{v}}$. The magnitude of $\vec{\mathbf{v}}$ is $|\vec{\mathbf{v}}| = \mathrm{abs}(\dot{x})$ is inherently non-negative.

2.3 Zero vector $\vec{0}$, a vector whose magnitude is zero

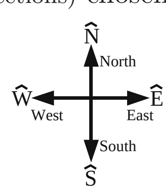
Addition with a zero vector:	$\mathbf{any}\vec{\mathbf{V}}\mathbf{ector} + \vec{\mathbf{0}} = \mathbf{any}\vec{\mathbf{V}}\mathbf{ector}$	
Dot product with a zero vector:	$\mathbf{any}\vec{\mathbf{V}}\mathbf{ector} \cdot \vec{\mathbf{0}} \underset{(2)}{=} 0$	$\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors
Cross product with a zero vector:	$\mathbf{any}\vec{\mathbf{V}}\mathbf{ector} \times \vec{\mathbf{0}} \underset{(5)}{=} \vec{\mathbf{0}}$	$\vec{\mathbf{0}}$ is ***parallel*** to all vectors
Derivative of the ***zero vector***:	$\frac{{}^{F}d\,\vec{\mathbf{0}}}{dt} = \vec{\mathbf{0}}$	F is any reference frame

Vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are said to be "***perpendicular***" if $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$ whereas $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" if $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$.
Note: Some say $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" only if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have the same direction and "anti-parallel" if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have opposite directions.[1]

2.4 Unit $\hat{\text{v}}$ectors, a vector whose magnitude is 1 (typeset with a special $\widehat{\text{hat}}$)

Unit vectors are "***sign posts***" (e.g., unit vectors $\widehat{\mathbf{N}}$, $\widehat{\mathbf{S}}$, $\widehat{\mathbf{W}}$, $\widehat{\mathbf{E}}$ for local Earth directions) chosen to simplify communication and calculations. Other useful "sign posts" are:

- Unit vector directed from one point to another point
- Unit vector directed locally vertical
- Unit vector tangent to a curve or perpendicular to a surface

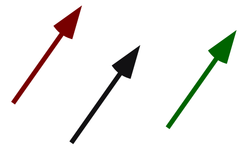


[1] The direction of a ***zero vector*** $\vec{\mathbf{0}}$ is arbitrary and may be regarded as having **any** direction so that $\vec{\mathbf{0}}$ is ***parallel*** to all vectors, $\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors, all zero vectors are equal, and one may use the definite pronoun "the" instead of the indefinite "a" e.g., "the zero vector". It is improper to say the ***zero vector*** has no direction as a vector is **defined** to have both magnitude and direction. It is also improper to say a ***zero vector*** has all directions as a vector is defined to have a magnitude and **a** direction.

A unit vector can be defined so it has the same direction as an arbitrary non-zero vector $\vec{\mathbf{v}}$ by dividing $\vec{\mathbf{v}}$ by $|\vec{\mathbf{v}}|$ (the magnitude of $\vec{\mathbf{v}}$). To avoid divide-by-zero problems during numerical computation, approximate the unit vector with a "small" positive real number ϵ in the denominator.

$$\textbf{unit}\widehat{\textbf{V}}\textbf{ector} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \approx \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}| + \epsilon} \quad (1)$$

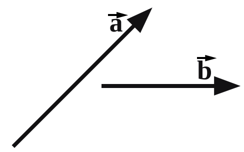
2.5 Equal vectors (=) vectors with the same magnitude and direction



Shown right are three ***equal vectors***. Although each has a different location, the vectors are equal because they have the same magnitude and direction.

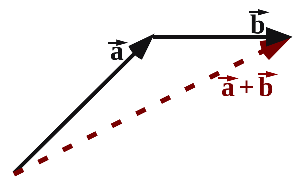
Some vectors have additional properties. For example, a position vector is associated with two points. Two position vectors are ***equal position vectors*** when, they have the same magnitude, same direction, and are associated with the same points. Two force vectors are ***equal force vectors*** when they have the same magnitude, direction, and point of application.

2.6 Vector addition (+)



As graphically shown right, adding two vectors $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ produces a vector. First, vector $\vec{\mathbf{b}}$ is translated[a] so its tail is at the tip of $\vec{\mathbf{a}}$. Next, the vector $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ is drawn from the tail of $\vec{\mathbf{a}}$ to the tip of the translated $\vec{\mathbf{b}}$.

Properties of vector addition



Commutative property: $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}}$

Associative property: $(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}$

Addition of zero vector: $\vec{\mathbf{a}} + \vec{\mathbf{0}} = \vec{\mathbf{a}}$

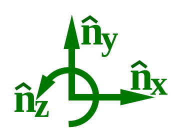
---

[a] Translating $\vec{\mathbf{b}}$ does *not* change the magnitude or direction of $\vec{\mathbf{b}}$, and so produces an equal $\vec{\mathbf{b}}$. Note: It is nonsensical to add vectors with different units, e.g., do not add a position vector with units of meters with a force vector with units of Newtons.

Example: Vector addition (+) algebra

Shown to the right is an example of how to add vector $\vec{\mathbf{w}}$ to vector $\vec{\mathbf{v}}$, each which is expressed in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.

$$\begin{aligned} \vec{\mathbf{v}} &= 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z \\ + \; \vec{\mathbf{w}} &= 2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z \\ \hline \vec{\mathbf{v}} + \vec{\mathbf{w}} &= 9\,\widehat{\mathbf{n}}_x + 8\,\widehat{\mathbf{n}}_y + 6\,\widehat{\mathbf{n}}_z \end{aligned}$$

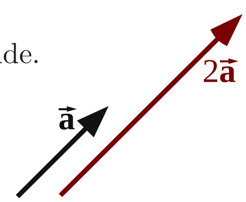


$$\vec{\mathbf{v}} = \underbrace{x\,\widehat{\mathbf{n}}_x}_{\text{vector component}} + \underbrace{y\,\widehat{\mathbf{n}}_y}_{\text{vector component}}$$

Special names for parts of the generic vector $\vec{\mathbf{v}}$.

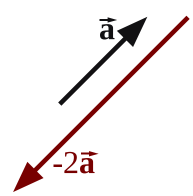
x is called the $\widehat{\mathbf{n}}_x$ ***scalar component*** (***measure***) of $\vec{\mathbf{v}}$.
y is called the $\widehat{\mathbf{n}}_y$ ***scalar component*** (***measure***) of $\vec{\mathbf{v}}$.

2.7 Vector multiplied or divided by a scalar ($*$ or /)



- Multiplying a vector by a **positive** number (other than 1) changes the vector's magnitude.
- Multiplying a vector by a **negative** number changes the vector's magnitude **and** reverses the ***sense*** of the vector.
- Dividing a vector $\vec{\mathbf{a}}$ by a scalar s is defined as $\frac{\vec{\mathbf{a}}}{s} \triangleq \frac{1}{s} * \vec{\mathbf{a}}$.

Properties of multiplication of a vector by a scalar s_1 or s_2



Commutative property: $s_1\,\vec{\mathbf{a}} = \vec{\mathbf{a}}\,s_1$

Associative property: $s_1\,(s_2\,\vec{\mathbf{a}}) = (s_1\,s_2)\,\vec{\mathbf{a}} = s_2\,(s_1\,\vec{\mathbf{a}}) = s_1\,s_2\,\vec{\mathbf{a}}$

Distributive property: $(s_1 + s_2)\,\vec{\mathbf{a}} = s_1\,\vec{\mathbf{a}} + s_2\,\vec{\mathbf{a}} \qquad s_1\,(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = s_1\,\vec{\mathbf{a}} + s_1\,\vec{\mathbf{b}}$

Multiplication by zero: $0 * \vec{\mathbf{a}} = \vec{\mathbf{0}}$

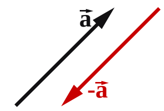
Example: Vector scalar multiplication and division ($*$ and /)

Given: $\vec{\mathbf{v}} = 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z$

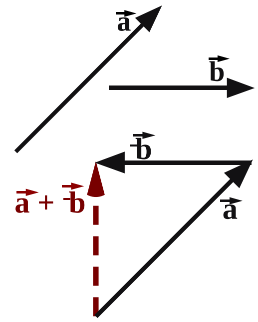
then: $5\,\vec{\mathbf{v}} = 35\,\widehat{\mathbf{n}}_x + 25\,\widehat{\mathbf{n}}_y + 20\,\widehat{\mathbf{n}}_z$

and $\frac{\vec{\mathbf{v}}}{-2} = -3.5\,\widehat{\mathbf{n}}_x - 2.5\,\widehat{\mathbf{n}}_y - 2\,\widehat{\mathbf{n}}_z$

2.8 Vector negation and subtraction (−)



Negation: As shown right, negating a vector (multiplying by −1) reverses the vector's ***sense*** (it points in the opposite direction). Negation does not change the vector's magnitude or orientation.



Subtraction: As the drawing to the right shows, subtracting a vector $\vec{b}$ from a vector $\vec{a}$ is simply addition and negation.[a]

$$\vec{a} - \vec{b} \triangleq \vec{a} + -\vec{b}$$

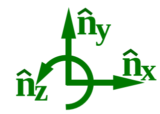
After negating vector $\vec{b}$, it is translated so the tail of $-\vec{b}$ is at the tip of $\vec{a}$. Next, vector $\vec{a} + -\vec{b}$ is drawn from the tail of $\vec{a}$ to the tip of the translated $-\vec{b}$.

[a] In most/all mathematics, subtraction is defined as negation and addition.

Example: Vector subtraction (−)

Shown right is an example of how to subtract vector $\vec{w}$ from vector $\vec{v}$, when each is expressed in terms of orthogonal unit vectors $\widehat{n}_x$, $\widehat{n}_y$, $\widehat{n}_z$.

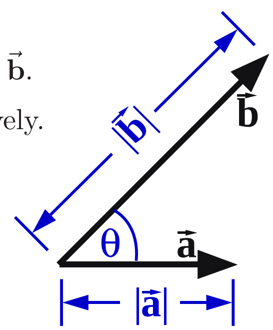
$$\begin{array}{rcl} \vec{v} &=& 7\,\widehat{n}_x + 5\,\widehat{n}_y + 4\,\widehat{n}_z \\ -\ \vec{w} &=& 2\,\widehat{n}_x + 3\,\widehat{n}_y + 2\,\widehat{n}_z \\ \hline \vec{v} - \vec{w} &=& 5\,\widehat{n}_x + 2\,\widehat{n}_y + 2\,\widehat{n}_z \end{array}$$



2.9 Vector dot product (·)

Equation (2) defines the ***dot product*** of vectors $\vec{a}$ and $\vec{b}$.

- $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of $\vec{a}$ and $\vec{b}$, respectively.
- θ is the smallest angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$.



$$\vec{a} \cdot \vec{b} \triangleq |\vec{a}|\,|\vec{b}|\cos(\theta) \qquad (2)$$

Equation (3) is a rearrangement of equation (2) that is useful for calculating the angle θ between two vectors.

$$\cos(\theta) \underset{(2)}{=} \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\;|\vec{b}|} \qquad (3)$$

Use **acos** to calculate θ.

Note: $\vec{a}$ and $\vec{b}$ are "***perpendicular***" when $\vec{a} \cdot \vec{b} = 0$.
Note: Dot-products encapsulate the ***law of cosines***.

Equation (2) shows $\vec{v} \cdot \vec{v} = |\vec{v}|^2$. Hence, the dot product can calculate a vector's ***magnitude*** as shown for $|\vec{v}|$ in equation (4).

Equation (4) also defines ***vector exponentiation*** $\vec{v}^n$ (vector $\vec{v}$ raised to scalar power n) as a non-negative scalar.

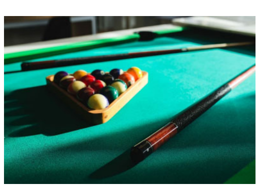
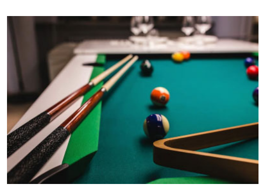
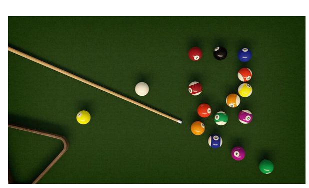
$$\begin{array}{rcl} \vec{v}^2 \triangleq |\vec{v}|^2 &=& \vec{v} \cdot \vec{v} \\ |\vec{v}| &=& +\sqrt{\vec{v} \cdot \vec{v}} \\ \vec{v}^n \triangleq |\vec{v}|^n &=& +(\vec{v} \cdot \vec{v})^{\frac{n}{2}} \end{array} \qquad (4)$$

Example: Kinetic energy $K = \frac{1}{2}\,m\,\vec{v}^2 \underset{(4)}{=} \frac{1}{2}\,m\,\vec{v} \cdot \vec{v}$

2.9.1 Properties of the dot-product (·)

Dot product with a zero vector	$\vec{a} \cdot \vec{0} = 0$	
Dot product of ***perpendicular*** vectors	$\vec{a} \cdot \vec{b} = 0$	if $\vec{a} \perp \vec{b}$
Dot product of parallel vectors	$\vec{a} \cdot \vec{b} = \pm\,|\vec{a}|\,|\vec{b}|$	if $\vec{a} \parallel \vec{b}$
Dot product with vectors scaled by s_1 and s_2	$s_1\,\vec{a} \cdot s_2\,\vec{b} = s_1\,s_2\,(\vec{a} \cdot \vec{b})$	
Commutative property	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	
Distributive property	$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$	
Distributive property	$(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$	

Note: The distributive property for dot-products and cross-products is proved in [33, pgs. 23-24, 32-34].

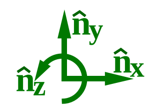
  

2.9.2 Uses for the dot-product ($\cdot$)

- Calculating an ***angle*** between two vectors [see equation (3) and example in Section 3.2]
- Determining when two vectors are ***perpendicular***, e.g., $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.
- Calculating a vector's ***magnitude*** [see equation (4) and ***distance*** examples in Sections 3.1 and 3.2].
- Changing a ***vector equation*** into a ***scalar equation*** (see Hw 2.29).
- Calculating a ***unit vector*** in the direction of a vector $\vec{\mathbf{v}}$ [from equation (1)] $\quad \textbf{unit}\vec{\textbf{Vector}} \underset{(1)}{=} \dfrac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$
- ***Projection*** of a vector $\vec{\mathbf{v}}$ in the direction of $\vec{\mathbf{b}}$, defined as: $\quad \vec{\mathbf{v}} \cdot \dfrac{\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|}$
  See Section 4.2 for ***projections***, ***measures***, ***coefficients***, ***components***.

2.9.3 Special case: Dot-products with orthogonal unit vectors

When $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are **orthogonal unit** vectors, it can be shown (see Hw 2.4)

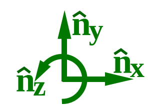


$$(a_x\,\widehat{\mathbf{n}}_x + a_y\,\widehat{\mathbf{n}}_y + a_z\,\widehat{\mathbf{n}}_z) \cdot (b_x\,\widehat{\mathbf{n}}_x + b_y\,\widehat{\mathbf{n}}_y + b_z\,\widehat{\mathbf{n}}_z) \;=\; a_x\,b_x + a_y\,b_y + a_z\,b_z$$

2.9.4 Examples: Vector dot-products ($\cdot$)

Shown below is how to use dot-products when vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are expressed in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.

$$\vec{\mathbf{v}} = 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z$$
$$\vec{\mathbf{w}} = 2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z$$



$\widehat{\mathbf{n}}_x$ measure of $\vec{\mathbf{v}}$ $\qquad \vec{\mathbf{v}} \cdot \widehat{\mathbf{n}}_x = 7$ $\quad$ (**measures** how much of $\vec{\mathbf{v}}$ is in the $\widehat{\mathbf{n}}_x$ direction).

$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 7^2 + 5^2 + 4^2 = 90 \qquad |\vec{\mathbf{v}}| = \sqrt{90} \approx 9.4868$

$\vec{\mathbf{w}} \cdot \vec{\mathbf{w}} = 2^2 + 3^2 + 2^2 = 17 \qquad |\vec{\mathbf{w}}| = \sqrt{17} \approx 4.1231$

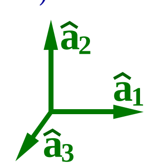
Unit vector in the direction of $\vec{\mathbf{v}}$: $\quad \dfrac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \dfrac{7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z}{\sqrt{90}} \approx 0.738\,\widehat{\mathbf{n}}_x + 0.527\,\widehat{\mathbf{n}}_y + 0.422\,\widehat{\mathbf{n}}_z$

Unit vector in the direction of $\vec{\mathbf{w}}$: $\quad \dfrac{\vec{\mathbf{w}}}{|\vec{\mathbf{w}}|} = \dfrac{2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z}{\sqrt{17}} \approx 0.485\,\widehat{\mathbf{n}}_x + 0.728\,\widehat{\mathbf{n}}_y + 0.485\,\widehat{\mathbf{n}}_z$

$\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 7*2 + 5*3 + 4*2 = 37 \qquad \angle(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = \text{acos}\left(\frac{37}{\sqrt{90}\,\sqrt{17}}\right) \approx 0.33 \text{ rad} \approx 18.93°$

2.9.5 Dot-products to change vector equations to scalar equations (see Hw 1.29)

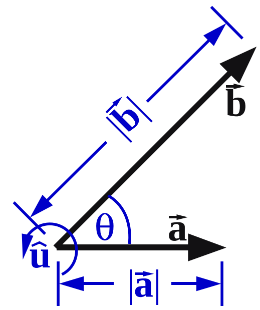
One way to form up to three linearly independent scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ is by dot-multiplying $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ with three orthogonal unit vectors $\widehat{\mathbf{a}}_1$, $\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_3$, i.e.,



$$\text{if} \quad \vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_1 = 0 \qquad \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_2 = 0 \qquad \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_3 = 0$$

Section 2.11.2 describes another way to form three ***different*** scalar equations from $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

2.10 Vector cross product ($\times$)

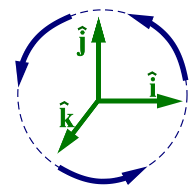


The ***cross product*** of a vector $\vec{\mathbf{a}}$ with a vector $\vec{\mathbf{b}}$ is defined in equation (5).

- $|\vec{\mathbf{a}}|$ and $|\vec{\mathbf{b}}|$ are the magnitudes of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, respectively
- θ is the smallest angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $(0 \le \theta \le \pi)$.
- $\hat{\mathbf{u}}$ is the unit vector **perpendicular** to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.
  The direction of $\hat{\mathbf{u}}$ is determined by the ***right-hand rule***.
  The right-hand rule is a convention like driving on the right-hand side of the road.
  Note: $|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\sin(\theta)$ [the coefficient of $\hat{\mathbf{u}}$ in equation (5)] is inherently non-negative because $\sin(\theta) \ge 0$ since $0 \le \theta \le \pi$. Hence, $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\sin(\theta)$.

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \triangleq |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\sin(\theta)\,\hat{\mathbf{u}} \qquad (5)$$

Properties of the cross-product ($\times$)



Cross product with a zero vector	$\vec{\mathbf{a}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}}$
Cross product of a vector with itself	$\vec{\mathbf{a}} \times \vec{\mathbf{a}} = \vec{\mathbf{0}}$
Cross product of ***parallel*** vectors	$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}} \quad \text{if } \vec{\mathbf{a}} \parallel \vec{\mathbf{b}}$
Cross product of scaled vectors	$s_1\,\vec{\mathbf{a}} \times s_2\,\vec{\mathbf{b}} = s_1\,s_2\,(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$
Distributive property	$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$
Cross products are ***not*** associative	$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) \neq (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}}$

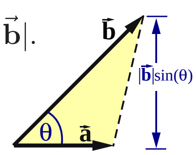
Cross products are ***not*** commutative
$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}} \qquad (6)$$

Vector triple cross product (bac-cab)
$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) \qquad (7)$$

A mnemonic for eqn (7) $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$ is "back cab" - as in were you born in the back of a cab?
Many proofs of this formula resolve $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ into orthogonal unit vectors (e.g., $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$) and equate components.

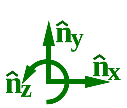
2.10.1 Uses for the cross-product ($\times$) in geometry, statics, motion analysis, ...

- ***Moment*** of a force, e.g., $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$
- ***Velocity/acceleration*** formulas, e.g., $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$ and $\vec{\mathbf{a}} = \vec{\boldsymbol{\alpha}} \times \vec{\mathbf{r}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$
- ***Perpendicular*** vectors, e.g., $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.
- ***Area of a triangle*** with sides of length $|\vec{\mathbf{a}}|$, $|\vec{\mathbf{b}}|$.
  Surveying: Hw 2.14 and Section 3.2 show how to use cross-products to calculate area.



$$\text{Vector area} \quad \vec{\boldsymbol{\Delta}}(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = \tfrac{1}{2}\;\vec{\mathbf{a}} \times \vec{\mathbf{b}}$$
$$\text{Scalar area} \quad \Delta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = \tfrac{1}{2}\;|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| \qquad (8)$$

2.10.2 Determinants and cross-products (with right-handed unit vectors)

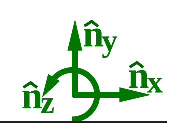


When $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ are **orthogonal unit** vectors, it can be shown (Hw 2.13) the cross product of two vectors happens to be equal to the ***determinant*** of an associated matrix.

$$\left.\begin{aligned} \vec{\mathbf{a}} &= a_x\,\hat{\mathbf{n}}_x + a_y\,\hat{\mathbf{n}}_y + a_z\,\hat{\mathbf{n}}_z \\ \vec{\mathbf{b}} &= b_x\,\hat{\mathbf{n}}_x + b_y\,\hat{\mathbf{n}}_y + b_z\,\hat{\mathbf{n}}_z \end{aligned}\right\} \qquad \vec{\mathbf{a}} \times \vec{\mathbf{b}} = \det\begin{bmatrix} \hat{\mathbf{n}}_x & \hat{\mathbf{n}}_y & \hat{\mathbf{n}}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = \begin{array}{l} \quad (a_y\,b_z - a_z\,b_y)\,\hat{\mathbf{n}}_x \\ -\;(a_x\,b_z - a_z\,b_x)\,\hat{\mathbf{n}}_y \\ +\;(a_x\,b_y - a_y\,b_x)\,\hat{\mathbf{n}}_z \end{array} \qquad (9)$$

Examples: Vector cross-products ($\times$) with determinants.

The following shows how to use cross-products with the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, each which is expressed in terms of the orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ shown to the right.



$$\left.\begin{array}{l}\vec{\mathbf{v}} \;=\; 7\,\widehat{\mathbf{n}}_{\rm x} \;+\; 5\,\widehat{\mathbf{n}}_{\rm y} \;+\; 4\,\widehat{\mathbf{n}}_{\rm z} \\ \vec{\mathbf{w}} \;=\; 2\,\widehat{\mathbf{n}}_{\rm x} \;+\; 3\,\widehat{\mathbf{n}}_{\rm y} \;+\; 2\,\widehat{\mathbf{n}}_{\rm z}\end{array}\right\} \qquad \vec{\mathbf{v}} \times \vec{\mathbf{w}} \;=\; \det\begin{bmatrix}\widehat{\mathbf{n}}_{\rm x} & \widehat{\mathbf{n}}_{\rm y} & \widehat{\mathbf{n}}_{\rm z} \\ 7 & 5 & 4 \\ 2 & 3 & 2\end{bmatrix} \;=\; \text{-}2\,\widehat{\mathbf{n}}_{\rm x} \;-\; 6\,\widehat{\mathbf{n}}_{\rm y} \;+\; 11\,\widehat{\mathbf{n}}_{\rm z}$$

Area from vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$: $\quad \Delta(\vec{\mathbf{v}}, \vec{\mathbf{w}}) \;=\; \frac{1}{2}\,|\vec{\mathbf{v}} \times \vec{\mathbf{w}}| \;=\; \frac{1}{2}\sqrt{(\text{-}2)^2 \;+\; (\text{-}6)^2 \;+\; 11^2} \;=\; \frac{\sqrt{161}}{2} \;\approx\; 6.344$

Scalar triple product: $\quad (2\,\widehat{\mathbf{n}}_{\rm x} + 3\,\widehat{\mathbf{n}}_{\rm y} + 4\,\widehat{\mathbf{n}}_{\rm z}) \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) \;=\; \det\begin{bmatrix}2 & 3 & 4 \\ 7 & 5 & 4 \\ 2 & 3 & 2\end{bmatrix} \;=\; 22$

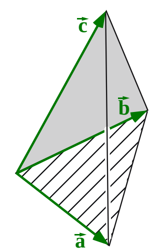
2.11 Optional: Scalar triple product $(\cdot\ \times$ or $\times\ \cdot)$

The ***scalar triple product*** of vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ is the scalar defined in the various ways shown below.

$$\text{ScalarTripleProduct} \;\triangleq\; \boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} \;=\; \vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}} \;=\; \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} \times \vec{\mathbf{a}} \;=\; \vec{\mathbf{b}} \times \vec{\mathbf{c}} \cdot \vec{\mathbf{a}} \qquad (10)$$

Although parentheses help clarify equation (10) e.g., $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ instead of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$, the parentheses are unnecessary because the cross product $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ **must** be performed before the dot product (for a sensible result to be produced).

2.11.1 Scalar triple product and the volume of a tetrahedron

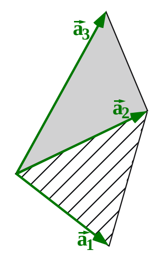


For a tetrahedron whose sides are described by the vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ (sides of length $|\vec{\mathbf{a}}|, |\vec{\mathbf{b}}|, |\vec{\mathbf{c}}|$), a geometrical interpretation of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$ is the ***volume of the parallelepiped***.

This formula helps calculate mass and volume (e.g., highway cut/fill calculations).

$$\text{Tetrahedron Volume} \;=\; \tfrac{1}{6}\;\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$$

2.11.2 $(\times\ \cdot)$ to change vector equations to scalar equations (see Hw 1.29)



Section 2.9.5 showed one method to form scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$. A 2^{nd} method expresses $\vec{\mathbf{v}}$ in terms of three non-coplanar (but not necessarily orthogonal or unit) vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3$, and writes the equally valid (but generally different) set of linearly independent scalar equations shown below.

Method 2: if $\;\vec{\mathbf{v}} \;=\; v_1\,\vec{\mathbf{a}}_1 \;+\; v_2\,\vec{\mathbf{a}}_2 \;+\; v_3\,\vec{\mathbf{a}}_3 \;=\; \vec{\mathbf{0}} \;\;\Rightarrow\;\; \boxed{v_1 = 0 \qquad v_2 = 0 \qquad v_3 = 0}$

Note: The proof that $v_i = 0 \;\; (i = 1, 2, 3)$ follows directly by substituting $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ into equation (4.2).

Courtesy Accuray Inc.. Vectors are widely useful, e.g., with medical robotics or cut/fill calculations for highway construction.