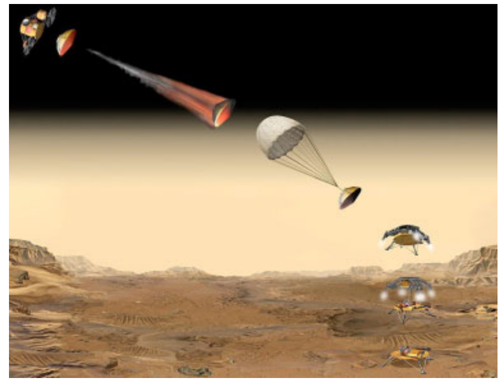


Chapter 2

Vectors ($\triangleq$ $\vec{a}+\vec{b}$ $-\vec{b}$ $\angle(\vec{a},\vec{b})$ $\vec{a}\cdot\vec{b}$ $\vec{a}\times\vec{b}$)

Examples in Hw 1, 2, 3

In 1881-1903, Gibbs developed ***vectors*** as a useful combination of magnitude and direction. Vectors are a very important **geometrical tool** (for surveying, motion, optics, graphics, CAD, FEA, etc.).

Symbol	Description	Details
$\vec{\mathbf{0}}$, $\hat{\mathbf{u}}$	Zero vector $\vec{\mathbf{0}}$ and unit vectors.	Sections 2.3, 2.4
$+\ -\ \ast\ /$	Vector addition, negation, subtraction, and scalar multiplication/division.	Sections 2.6 - 2.8
$\cdot\ \ \times$	Vector dot product and cross product.	Sections 2.9, 2.10
$\frac{^F d}{dt}$	Vector differentiation.	Chapters 6, 7

2.1 Examples of scalars vectors and dyadics

- A ***scalar*** is a number, possibly with units (e.g., 7 $\frac{\text{m}}{\text{s}}$ or 9 kg), such as

time	density	volume	mass	potential energy	work
distance	speed	angle	weight	kinetic energy	temperature

- A ***vector*** is a quantity with magnitude and ***one*** associated direction (e.g., $7\,\widehat{\mathbf{up}}$). For example, a ***velocity vector*** has speed (how fast something moves) and direction (which way it is going). A ***force vector*** has magnitude (how hard something is pushed) and direction (which way it is shoved). Examples include:

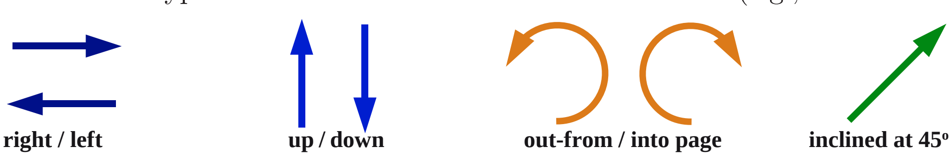
force	velocity	acceleration	translational momentum
torque	angular velocity	angular acceleration	angular momentum

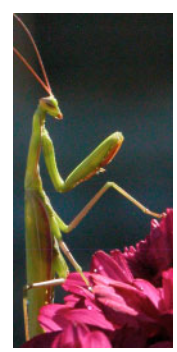
In 1884, Gibbs re-defined ***vectors*** and taught them with **90** lectures.

- A ***dyad*** is a quantity with magnitude and ***two*** associated directions (e.g., $8\,\hat{\mathbf{i}}\,\hat{\mathbf{j}}$). For example, ***stress*** associates with area and force (both regarded as vectors). A ***dyadic*** is the **sum of dyads** (Chapter 13), e.g., an ***inertia dyadic*** (Chapter 14) is the sum of dyads associated with moments and products of inertia.

2.2 Definition of a vector

A ***vector*** is defined as a quantity having ***magnitude*** and ***direction***.[a]
Vectors are represented pictorially with straight or curved arrows (examples below).
Vectors are typeset with **bold font** and an $\vec{\mathbf{a}}$rrow or $\hat{\mathbf{h}}$at (e.g., $\vec{\mathbf{v}}$ denotes a vector).





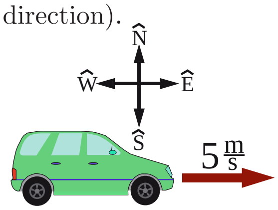
Courtesy Bro. Claude Rheaume. LaSalette.

Certain vectors have additional properties, e.g., a ***position vector*** $\vec{\mathbf{r}}$ has two associated points and units of length (e.g., meters) and a ***unit vector*** has magnitude 1 (no units).

[a] A vector's ***magnitude*** is a real non-negative scalar (e.g., 7 m/s). A vector's ***direction*** is its ***orientation*** and ***sense***. A vector is similar to a **ray** in direction, but a vector has finite magnitude. A vector is similar to a

line segment in magnitude and orientation, but a vector also has a ***sense*** (a fully defined direction).

Example of a vector: Consider the statement "the car is moving East at 5 $\frac{\text{m}}{\text{s}}$". It is convenient to represent the car's speed and direction with the velocity vector $\vec{\mathbf{v}} = 5\,\widehat{\mathbf{East}}$ (a hat designates the direction $\widehat{\mathbf{East}}$ as a ***unit vector***). The car's speed is always a real non-negative scalar denoted $|\vec{\mathbf{v}}|$ (the ***magnitude*** of $\vec{\mathbf{v}}$). The combination of ***magnitude*** and ***direction*** is a ***vector***.



The velocity of a car with speed 5 $\frac{\text{m}}{\text{s}}$ moving West can also be written as $\vec{\mathbf{v}} = \text{-}5\,\widehat{\mathbf{East}}$. The negative sign in $\text{-}5\,\widehat{\mathbf{East}}$ reverses vector $\vec{\mathbf{v}}$'s direction whereas $\vec{\mathbf{v}}$'s magnitude is $|\vec{\mathbf{v}}| = |\text{-}5\,\widehat{\mathbf{East}}| = 5\,\frac{\text{m}}{\text{s}}$.

Note: When a vector $\vec{\mathbf{v}}$ is written $\vec{\mathbf{v}} = v\,\widehat{\mathbf{East}}$ v is called the $\widehat{\mathbf{East}}$ ***measure*** of vector $\vec{\mathbf{v}}$ and is a **negative**, **zero**, or **positive** real scalar. The magnitude of $\vec{\mathbf{v}}$ is $|\vec{\mathbf{v}}| = \text{abs}(v)$ is inherently non-negative.

2.3 Zero vector $\vec{0}$, a vector whose magnitude is zero

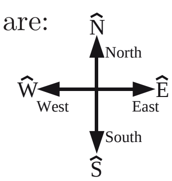
Addition with $\vec{\mathbf{0}}$	$\text{any}\vec{\text{V}}\text{ector} + \vec{\mathbf{0}}$	$= \text{any}\vec{\text{V}}\text{ector}$	
Dot product with $\vec{\mathbf{0}}$	$\text{any}\vec{\text{V}}\text{ector} \cdot \vec{\mathbf{0}}$	$\underset{(2)}{=} 0$	$\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors.
Cross product with $\vec{\mathbf{0}}$	$\text{any}\vec{\text{V}}\text{ector} \times \vec{\mathbf{0}}$	$\underset{(5)}{=} \vec{\mathbf{0}}$	$\vec{\mathbf{0}}$ is ***parallel*** to all vectors.

Vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are said to be "***perpendicular***" if $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$ whereas $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" if $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$.
Some say $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are ***parallel*** if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have the same direction and ***anti-parallel*** if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have opposite directions.
The direction of $\vec{\mathbf{0}}$ is arbitrary and may be regarded as having **any** direction, hence $\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors, $\vec{\mathbf{0}}$ is ***parallel*** and ***anti-parallel*** to all vectors, and all zero vectors are equal. It is improper to say $\vec{\mathbf{0}}$ has no direction as a vector is **defined** to have both magnitude **and** direction. The ***zero scalar*** 0 has 0 magnitude and **no** direction, whereas the ***zero vector*** $\vec{\mathbf{0}}$ has a direction (albeit undefined).

2.4 Unit $\hat{\text{v}}$ectors: Vectors with magnitude 1 and no units (typeset with a $\hat{\text{h}}$at)

Unit vectors are "***sign posts***" (e.g., unit vectors $\widehat{\mathbf{N}}$, $\widehat{\mathbf{S}}$, $\widehat{\mathbf{W}}$, $\widehat{\mathbf{E}}$ $\widehat{\mathbf{up}}$ for local Earth directions) that simplify communication and calculations. Other useful "sign posts" are:

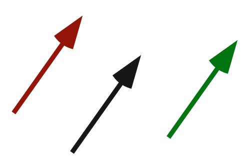
- Unit vector directed from one point to another point
- Unit vector directed locally vertical
- Unit vector tangent to a curve or perpendicular to a surface



A unit vector can be defined so it has the same direction as an arbitrary non-zero vector $\vec{\mathbf{v}}$ by dividing $\vec{\mathbf{v}}$ by $|\vec{\mathbf{v}}|$ (the magnitude of $\vec{\mathbf{v}}$). To avoid divide-by-zero problems in numerical computation, approximate the unit vector with a "small" positive real number ϵ in the denominator.

$$\mathbf{unit}\widehat{\mathbf{V}}\mathbf{ector} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \approx \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}| + \epsilon} \tag{1}$$

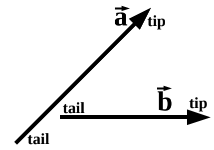
2.5 Equal vectors (=) vectors with the same magnitude and direction



Shown right are three ***equal vectors***. Although each has a different location, the vectors are equal because they have the same magnitude and direction.

Some vectors have additional properties. For example, a position vector has 2 associated points. Two position vectors are ***equal position vectors*** if they have the same magnitude, same direction, **and** same 2 associated points. Two force vectors are ***equal force vectors*** if they have the same magnitude, direction, **and** same point of application.

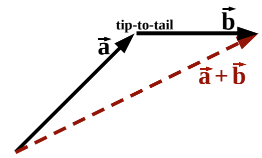
2.6 Vector addition (+)



As shown right, adding vectors $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ produces a vector. First $\vec{\mathbf{b}}$ is translated so its ***tail*** is at the ***tip*** of $\vec{\mathbf{a}}$. Next, $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ is drawn from the ***tail*** of $\vec{\mathbf{a}}$ to the ***tip*** of the translated $\vec{\mathbf{b}}$.

Translating $\vec{\mathbf{b}}$ does ***not*** change $\vec{\mathbf{b}}$'s magnitude or direction, and so produces an equal $\vec{\mathbf{b}}$.

Properties of vector addition



Commutative property: $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}}$

Associative property: $(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}$

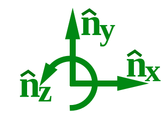
Addition of zero vector: $\vec{\mathbf{a}} + \vec{\mathbf{0}} = \vec{\mathbf{a}}$

Vectors with different units do **not** add. Do **not** add a position vector (units of meters) with a force vector (units of Newtons). Note: A scalar cannot be added to a vector, e.g., $5 + \vec{\mathbf{v}}$ does not make sense.

Example: Vector addition (+)

Shown right is how to add vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, each of which is expressed in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.

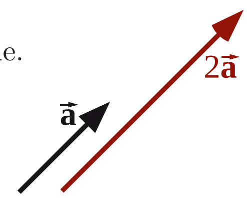
$$
\begin{aligned}
\vec{\mathbf{v}} &= 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z \\
-\ \vec{\mathbf{w}} &= 2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z \\
\hline
\vec{\mathbf{v}} + \vec{\mathbf{w}} &= 9\,\widehat{\mathbf{n}}_x + 8\,\widehat{\mathbf{n}}_y + 6\,\widehat{\mathbf{n}}_z
\end{aligned}
$$



$$\vec{\mathbf{v}} = \underbrace{x\,\widehat{\mathbf{n}}_x}_{\text{vector component}} + \underbrace{y\,\widehat{\mathbf{n}}_y}_{\text{vector component}}$$

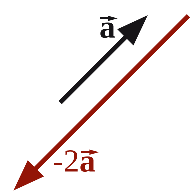
Special names for parts of the generic vector $\vec{\mathbf{v}}$.
x is called the $\widehat{\mathbf{n}}_x$ ***measure*** (or ***scalar component***) of $\vec{\mathbf{v}}$.
y is called the $\widehat{\mathbf{n}}_y$ ***measure*** (or ***scalar component***) of $\vec{\mathbf{v}}$.

2.7 Vector multiplied or divided by a scalar ($*$ or /)



- Multiplying a vector by a **positive** number (other than 1) changes the vector's magnitude.
- Multiplying a vector by a **negative** number changes the vector's magnitude **and** reverses the ***sense*** of the vector.
- Dividing a vector $\vec{\mathbf{a}}$ by a scalar s is defined as $\frac{\vec{\mathbf{a}}}{s} \triangleq \frac{1}{s} * \vec{\mathbf{a}}$.

Properties of multiplication of a vector by a scalar s_1 or s_2



Commutative property: $s_1\,\vec{\mathbf{a}} = \vec{\mathbf{a}}\,s_1$

Associative property: $s_1\,(s_2\,\vec{\mathbf{a}}) = (s_1\,s_2)\,\vec{\mathbf{a}} = s_2\,(s_1\,\vec{\mathbf{a}}) = s_1\,s_2\,\vec{\mathbf{a}}$

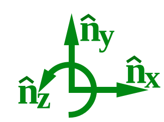
Distributive property: $(s_1 + s_2)\,\vec{\mathbf{a}} = s_1\,\vec{\mathbf{a}} + s_2\,\vec{\mathbf{a}}$ $\qquad s_1\,(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = s_1\,\vec{\mathbf{a}} + s_1\,\vec{\mathbf{b}}$

Multiplication by zero: $0 * \vec{\mathbf{a}} = \vec{\mathbf{0}}$

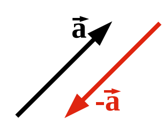
Example: Vector scalar multiplication and division ($*$ and /)

Given: $\vec{\mathbf{v}} = 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z$
then: $5\,\vec{\mathbf{v}} = 35\,\widehat{\mathbf{n}}_x + 25\,\widehat{\mathbf{n}}_y + 20\,\widehat{\mathbf{n}}_z$

and $\frac{-\vec{\mathbf{v}}}{2} = -3.5\,\widehat{\mathbf{n}}_x - 2.5\,\widehat{\mathbf{n}}_y - 2\,\widehat{\mathbf{n}}_z$



2.8 Vector negation and subtraction (−)

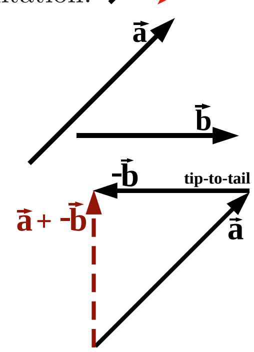


Negation: As shown right, negating a vector (multiplying by -1) reverses the vector's ***sense*** (it points in the opposite direction). Negation does not change the vector's magnitude or orientation.

Subtraction: As the drawing to the right shows, subtracting a vector $\vec{\mathbf{b}}$ from a vector $\vec{\mathbf{a}}$ is simply addition and negation.

$$\vec{\mathbf{a}} - \vec{\mathbf{b}} \triangleq \vec{\mathbf{a}} + -\vec{\mathbf{b}}$$

Note: In most/all mathematics, subtraction is defined as negation and addition.



After negating vector $\vec{\mathbf{b}}$, it is translated so the ***tail*** of $-\vec{\mathbf{b}}$ is at the ***tip*** of $\vec{\mathbf{a}}$.
Next, vector $\vec{\mathbf{a}} + -\vec{\mathbf{b}}$ is drawn from the ***tail*** of $\vec{\mathbf{a}}$ to the ***tip*** of the translated $-\vec{\mathbf{b}}$.

Example: Vector subtraction ($\vec{\mathbf{v}} - \vec{\mathbf{w}}$)

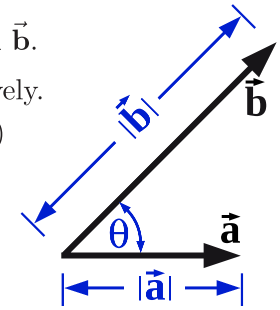
It is easy to subtract vectors that are expressed in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.

$$
\begin{aligned}
\vec{\mathbf{v}} &= 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z \\
\vec{\mathbf{w}} &= 2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z \\
\hline
\vec{\mathbf{v}} - \vec{\mathbf{w}} &= 5\,\widehat{\mathbf{n}}_x + 2\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z
\end{aligned}
$$

2.9 Vector dot product $(\cdot)$

Equation (2) defines the ***dot product*** of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

- $|\vec{\mathbf{a}}|$ and $|\vec{\mathbf{b}}|$ are the magnitudes of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, respectively.
- θ is the smallest angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $(0 \le \theta \le \pi)$
  To visualize θ, draw $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as ***tail-to-tail***.



$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \triangleq |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\cos(\theta) \qquad (2)$$

Equation (3) is a rearrangement of equation (2) that is useful for calculating the angle θ between two vectors.
Note: $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***perpendicular***" when $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.
Note: Dot-products encapsulate the ***law of cosines***.

$$\cos(\theta) \underset{(2)}{=} \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|\;|\vec{\mathbf{b}}|} \qquad (3)$$

Use **acos** to calculate θ.

Equation (2) shows $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{v}}|^2$. Hence, the dot product can calculate a vector's ***magnitude*** as shown for $|\vec{\mathbf{v}}|$ in equation (4).

Equation (4) also defines ***vector exponentiation*** $\vec{\mathbf{v}}^n$ (vector $\vec{\mathbf{v}}$ raised to scalar power n) as a non-negative scalar.

Example: Kinetic energy $\text{K} = \frac{1}{2}\,\text{m}\,\vec{\mathbf{v}}^2 \underset{(4)}{=} \frac{1}{2}\,\text{m}\,\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$

$$\begin{aligned} \vec{\mathbf{v}}^2 \triangleq |\vec{\mathbf{v}}|^2 &= \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \\ |\vec{\mathbf{v}}| &= {}^+\sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}} \\ \vec{\mathbf{v}}^n \triangleq |\vec{\mathbf{v}}|^n &= {}^+(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})^{\frac{n}{2}} \end{aligned} \qquad (4)$$

2.9.1 Properties of the dot-product $(\cdot)$

Dot product with a zero vector	$\vec{\mathbf{a}} \cdot \vec{\mathbf{0}} = 0$	
Dot product of ***perpendicular*** vectors	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$	if $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$
Dot product of parallel vectors	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \pm\,|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|$	if $\vec{\mathbf{a}} \parallel \vec{\mathbf{b}}$
Dot product with vectors scaled by s_1 and s_2	$s_1\,\vec{\mathbf{a}} \cdot s_2\,\vec{\mathbf{b}} = s_1\,s_2\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$	
Commutative property	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$	
Distributive property	$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$	
Distributive property	$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot (\vec{\mathbf{c}} + \vec{\mathbf{d}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{d}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{d}}$	

Note: The distributive property for dot-products and cross-products is proved in [34, pgs. 23-24, 32-34].

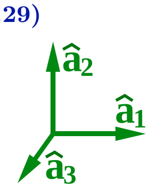
2.9.2 Uses for the dot-product $(\cdot)$

- Determining the ***angle*** between two vectors [see equation (3) and example in Section 3.3].
- Determining when two vectors are ***perpendicular***, e.g., $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.
- Calculating a vector's ***magnitude*** [see equation (4) and ***distance*** examples in Sections 3.1 and 3.3].
- Changing a ***vector equation*** into a ***scalar equation*** (see Hw 2.29).
- Calculating a ***unit vector*** in the direction of a vector $\vec{\mathbf{v}}$ [from equation (1)] $\quad \textbf{unit}\vec{\textbf{V}}\textbf{ector} \underset{(1)}{=} \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$
- ***Projection*** of a vector $\vec{\mathbf{v}}$ in the direction of $\vec{\mathbf{b}}$, defined as: $\quad \vec{\mathbf{v}} \cdot \frac{\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|}$
  See Section 4.2 for ***projections***, ***measures***, ***coefficients***, ***components***.
  See Section 3.3 for a distance measure from a point to a plane.

2.9.3 Dot-products to change vector equations to scalar equations (see Hw 1.29)

One way to form up to three linearly independent scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ is by dot-multiplying $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ with three orthogonal unit vectors $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, $\hat{\mathbf{a}}_3$, i.e.,

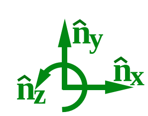
$$\text{if} \quad \vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \vec{\mathbf{v}} \cdot \hat{\mathbf{a}}_1 = 0 \qquad \vec{\mathbf{v}} \cdot \hat{\mathbf{a}}_2 = 0 \qquad \vec{\mathbf{v}} \cdot \hat{\mathbf{a}}_3 = 0$$



2.9.4 Special case: Dot-products with orthogonal unit vectors

When $\hat{\mathbf{n}}_\text{x}$, $\hat{\mathbf{n}}_\text{y}$, $\hat{\mathbf{n}}_\text{z}$ are **orthogonal unit** vectors, it can be shown (see Hw 2.4)

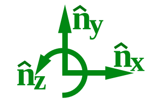
$$(a_x\,\hat{\mathbf{n}}_\text{x} + a_y\,\hat{\mathbf{n}}_\text{y} + a_z\,\hat{\mathbf{n}}_\text{z}) \cdot (b_x\,\hat{\mathbf{n}}_\text{x} + b_y\,\hat{\mathbf{n}}_\text{y} + b_z\,\hat{\mathbf{n}}_\text{z}) = a_x\,b_x + a_y\,b_y + a_z\,b_z$$



2.9.5 Examples: Vector dot-products ($\cdot$)

Shown below is how to use dot-products when vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are expressed in terms of orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$.

$\vec{\mathbf{v}} = 7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 4\,\hat{\mathbf{n}}_z$
$\vec{\mathbf{w}} = 2\,\hat{\mathbf{n}}_x + 3\,\hat{\mathbf{n}}_y + 2\,\hat{\mathbf{n}}_z$

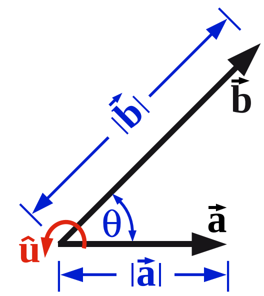


$\hat{\mathbf{n}}_x$ measure of $\vec{\mathbf{v}}$	$\vec{\mathbf{v}} \cdot \hat{\mathbf{n}}_x = 7$ (**measures** how much of $\vec{\mathbf{v}}$ is in the $\hat{\mathbf{n}}_x$ direction).
$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 7^2 + 5^2 + 4^2 = 90$	$\lvert\vec{\mathbf{v}}\rvert = \sqrt{90} \approx 9.4868$
$\vec{\mathbf{w}} \cdot \vec{\mathbf{w}} = 2^2 + 3^2 + 2^2 = 17$	$\lvert\vec{\mathbf{w}}\rvert = \sqrt{17} \approx 4.1231$
Unit vector in the direction of $\vec{\mathbf{v}}$:	$\frac{\vec{\mathbf{v}}}{\lvert\vec{\mathbf{v}}\rvert} = \frac{7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 4\,\hat{\mathbf{n}}_z}{\sqrt{90}} \approx 0.738\,\hat{\mathbf{n}}_x + 0.527\,\hat{\mathbf{n}}_y + 0.422\,\hat{\mathbf{n}}_z$
Unit vector in the direction of $\vec{\mathbf{w}}$:	$\frac{\vec{\mathbf{w}}}{\lvert\vec{\mathbf{w}}\rvert} = \frac{2\,\hat{\mathbf{n}}_x + 3\,\hat{\mathbf{n}}_y + 2\,\hat{\mathbf{n}}_z}{\sqrt{17}} \approx 0.485\,\hat{\mathbf{n}}_x + 0.728\,\hat{\mathbf{n}}_y + 0.485\,\hat{\mathbf{n}}_z$
$\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 7*2 + 5*3 + 4*2 = 37$	$\angle(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = \text{acos}(\frac{37}{\sqrt{90}\,\sqrt{17}}) \approx 0.33 \text{ rad} \approx 18.93°$

2.10 Vector cross product ($\times$)

The ***cross product*** of a vector $\vec{\mathbf{a}}$ with a vector $\vec{\mathbf{b}}$ is defined in equation (5).

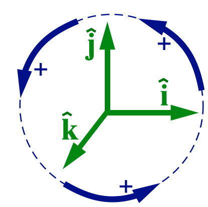
- $\lvert\vec{\mathbf{a}}\rvert$ and $\lvert\vec{\mathbf{b}}\rvert$ are the magnitudes of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, respectively
- θ is the smallest angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $(0 \leq \theta \leq \pi)$.
  To visualize θ, draw $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as ***tail-to-tail***.
- $\hat{\mathbf{u}}$ is the unit vector **perpendicular** to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.
  The direction of $\hat{\mathbf{u}}$ is determined by the ***right-hand rule***.
  The right-hand rule is a convention like driving on the right-hand side of the road.
  Note: $\lvert\vec{\mathbf{a}}\rvert\,\lvert\vec{\mathbf{b}}\rvert\,\sin(\theta)$ [the coefficient of $\hat{\mathbf{u}}$ in equation (5)] is inherently non-negative because $\sin(\theta) \geq 0$ since $0 \leq \theta \leq \pi$. Hence, $\lvert\vec{\mathbf{a}} \times \vec{\mathbf{b}}\rvert = \lvert\vec{\mathbf{a}}\rvert\,\lvert\vec{\mathbf{b}}\rvert\,\sin(\theta)$.



$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \triangleq \lvert\vec{\mathbf{a}}\rvert\,\lvert\vec{\mathbf{b}}\rvert\,\sin(\theta)\,\hat{\mathbf{u}} \qquad (5)$$

$\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is $\perp$ to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Properties of the cross-product ($\times$)



Cross product with a zero vector	$\vec{\mathbf{a}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}}$	
Cross product of a vector with itself	$\vec{\mathbf{a}} \times \vec{\mathbf{a}} = \vec{\mathbf{0}}$	
Cross product of ***parallel*** vectors	$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$	if $\vec{\mathbf{a}} \parallel \vec{\mathbf{b}}$
Cross product of scaled vectors	$s_1\,\vec{\mathbf{a}} \times s_2\,\vec{\mathbf{b}} = s_1\,s_2\,(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$	
Distributive property	$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$	
Cross products are ***not*** associative	$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) \neq (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}}$	
Cross products are ***not*** commutative.	$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \text{-}\vec{\mathbf{b}} \times \vec{\mathbf{a}}$	(6)
Vector triple cross product (bac-cab).	$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$	(7)

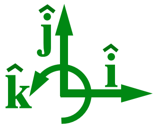
A mnemonic for eqn (7) is "back cab" - as in were you born in the back of a cab? Many proofs of this formula resolve $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ into orthogonal unit vectors (e.g., $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$) and equate components.

2.10.1 Uses for the cross-product ($\times$) in geometry, statics, motion analysis, ...

- ***Moment*** of a force such as $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$ (details in Section 17.1).
- ***Velocity/acceleration*** formulas [see eqns (8.3, 8.4)] $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$ and $\vec{\mathbf{a}} = \vec{\boldsymbol{\alpha}} \times \vec{\mathbf{r}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$.
- ***Perpendicular*** vectors, e.g., $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.
- ***Area of a triangle*** with sides $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ (see Sections 3.2, 3.3 and Hw 2.13). $\vec{\boldsymbol{\Delta}}(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = \frac{1}{2}\,\vec{\mathbf{a}} \times \vec{\mathbf{b}}$.

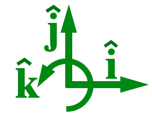
2.10.2 Determinants and cross-products (with right-handed unit vectors)

When vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are expressed in terms of **orthogonal unit** vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$, it can be shown (Hw 2.12) that $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ happens to equal the ***determinant*** of an associated matrix.



$$\left.\begin{aligned}\vec{\mathbf{a}} &= a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}}\\ \vec{\mathbf{b}} &= b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\,\hat{\mathbf{k}}\end{aligned}\right\} \qquad \vec{\mathbf{a}} \times \vec{\mathbf{b}} = \det\begin{bmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{bmatrix} = \begin{array}{l}\ (a_y\,b_z - a_z\,b_y)\,\hat{\mathbf{i}}\\ -\ (a_x\,b_z - a_z\,b_x)\,\hat{\mathbf{j}}\\ +\ (a_x\,b_y - a_y\,b_x)\,\hat{\mathbf{k}}\end{array} \tag{8}$$

Examples: Vector cross-products (×) with determinants.
The following shows how to use cross-products with the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, each which is expressed in terms of the orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ shown to the right.



$$\left.\begin{aligned}\vec{\mathbf{v}} &= 7\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}\\ \vec{\mathbf{w}} &= 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}\end{aligned}\right\} \qquad \vec{\mathbf{v}} \times \vec{\mathbf{w}} = \det\begin{bmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 7 & 5 & 4\\ 2 & 3 & 2\end{bmatrix} = \text{-}2\,\hat{\mathbf{i}} - 6\,\hat{\mathbf{j}} + 11\,\hat{\mathbf{k}}$$

$$\text{Scalar triple product:} \quad (2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}) \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \det\begin{bmatrix}2 & 3 & 4\\ 7 & 5 & 4\\ 2 & 3 & 2\end{bmatrix} = 22$$

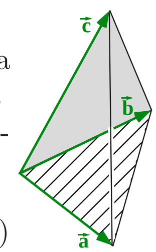
2.11 Optional: Scalar triple product (· × or × ·)

The ***scalar triple product*** of vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ is the scalar defined in the various ways shown below.

$$\text{ScalarTripleProduct} \triangleq \boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} \times \vec{\mathbf{a}} = \vec{\mathbf{b}} \times \vec{\mathbf{c}} \cdot \vec{\mathbf{a}} \tag{9}$$

Although parentheses clarify eqn (9) e.g., $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ instead of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$, the parentheses are unnecessary because the cross product $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ **must** be performed before the dot product (for a sensible result to be produced).
Note: $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{a}}) = 0$ since [by eqn (5)] $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is ⊥ to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and the dot-product of $\vec{\mathbf{a}}$ with a vector ⊥ to $\vec{\mathbf{a}}$ is 0.

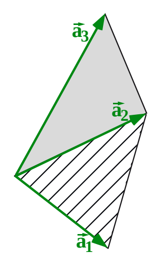
2.11.1 Scalar triple product and the volume of a tetrahedron



For a tetrahedron whose sides are described by the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ (sides of length $|\vec{\mathbf{a}}|$, $|\vec{\mathbf{b}}|$, $|\vec{\mathbf{c}}|$), a geometrical interpretation of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$ is the ***volume of the parallelepiped***. This formula helps calculate mass and volume of generic 3D shapes (e.g., for highway cut/fill calculations and CAD solid modeling). A tetrahedron's volume is calculated in Section 3.3.

$$\boxed{\text{Tetrahedron Volume} = \frac{1}{6}\,\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} = \frac{1}{6}\,\vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} \underset{(3.3)}{=} \frac{1}{3}\,\vec{\boldsymbol{\Delta}}(\vec{\mathbf{a}}, \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}} \tag{10}$$

2.11.2 (× ·) to change vector equations to scalar equations (see Hw 1.29)



Section 2.9.3 showed one method to form scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$. A 2^{nd} method expresses $\vec{\mathbf{v}}$ in terms of three non-coplanar (but not necessarily orthogonal or unit) vectors $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, and writes the equally valid (but generally different) set of linearly independent scalar equations shown below [proved by directly by substituting $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ into eqn (4.2)].

$$\text{Method 2: if} \quad \vec{\mathbf{v}} = v_1\,\vec{\mathbf{a}}_1 + v_2\,\vec{\mathbf{a}}_2 + v_3\,\vec{\mathbf{a}}_3 = \vec{\mathbf{0}} \quad \Rightarrow \quad \boxed{v_1 = 0 \qquad v_2 = 0 \qquad v_3 = 0}$$

Courtesy Accuray Inc. Vectors are widely useful, e.g., in medical robotics, cut/fill calculations for highway & railway construction, ...