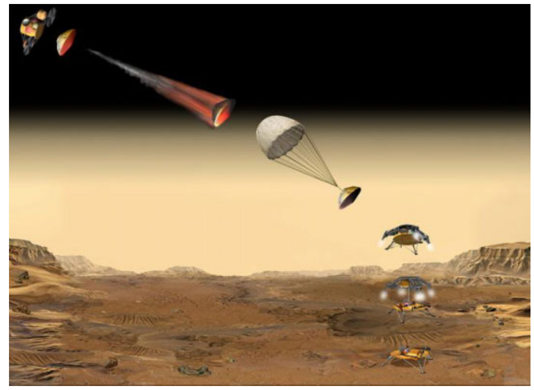
Chapter 2

Vectors



Courtesy NASA/JPL-Caltech

Summary (see examples in Hw 1, 2, 3)

Circa 1900 A.D., J. Williard Gibbs invented a useful combination of magnitude and direction called ***vectors*** and their higher-dimensional counterparts ***dyadics***, ***triadics***, and ***polyadics***. Vectors are an important **geometrical tool** e.g., for surveying, motion analysis, lasers, optics, computer graphics, animation, CAD/CAE (computer aided drawing/engineering), and FEA.

Symbol	Description	Details
$\vec{\mathbf{0}}$, $\hat{\mathbf{u}}$	Zero vector and unit vector.	Sections 2.3, 2.4
$+ \; - \; *$	Vector addition, negation, subtraction, and multiplication/division with a scalar.	Sections 2.6 - 2.8
$\cdot \quad \times$	Vector dot product and cross product.	Sections 2.9, 2.10
${}^{F}\frac{d}{dt}$	Vector differentiation.	Chapters 6, 7

2.1 Examples of scalars, vectors, and dyadics

- A ***scalar*** is a non-directional quantity (e.g., a real number). Examples include:

time	density	volume	mass	moment of inertia	temperature
distance	speed	angle	weight	potential energy	kinetic energy

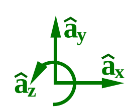
- A ***vector*** is a quantity that has magnitude and ***one*** associated direction. For example, a ***velocity vector*** has speed (how fast something is moving) and direction (which way it is going). A ***force vector*** has magnitude (how hard something is pushed) and direction (which way it is shoved). Examples include:

position vector	velocity	acceleration	translational momentum	force
impulse	angular velocity	angular acceleration	angular momentum	torque

- A ***dyad*** is a quantity with magnitude and ***two*** associated directions. For example, ***stress*** associates with area and force (both regarded as vectors). A ***dyadic*** is the **sum of dyads**. For example, an ***inertia dyadic*** (Chapter 14) is the sum of dyads associated with moments and products of inertia.

Words: Vector and column matrices. Although mathematics uses the word[1] "vector" to describe a column matrix, a column matrix does ***not*** have direction. To associate direction, attach a basis e.g., as shown below.

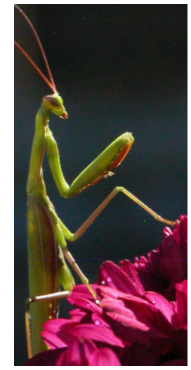
$$\hat{\mathbf{a}}_x + 2\,\hat{\mathbf{a}}_y + 3\,\hat{\mathbf{a}}_z = \begin{bmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\hat{\mathbf{a}}_{xyz}} \quad \text{where } \hat{\mathbf{a}}_x,\, \hat{\mathbf{a}}_y,\, \hat{\mathbf{a}}_z \text{ are orthogonal unit vectors}$$



Note: Although it can be helpful to represent vectors with orthogonal unit vectors (e.g., $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ or $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$), it is not always necessary, desirable, or efficient. Postponing resolution of vectors into components allows maximum use of simplifying vector properties and avoids simplifications such as $\sin^2(\theta) + \cos^2(\theta) = 1$ (see Homework 2.9).

[1] Words have context. Some words are contranyms (opposite meanings) such as "fast" and "bolt" (move quickly or fasten).

2.2 Definition of a vector



Courtesy Bro. Claude Rheaume LaSalette.

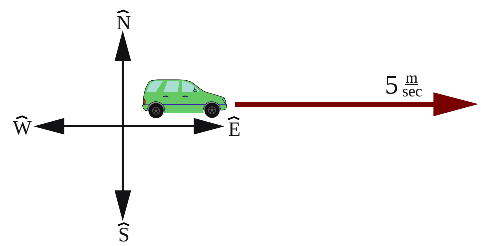
A ***vector*** is defined as a quantity having ***magnitude*** and ***direction***.[a]
Vectors are represented graphically with straight or curved arrows (examples below).

Certain vectors have additional special properties. For example, a ***position vector*** is associated with two points and has units of distance. A ***bound vector*** such as ***force*** is associated with a point (or line of action).

[a] A vector's ***magnitude*** is a real non-negative number. A vector's ***direction*** can be resolved into ***orientation*** and ***sense***. For example, a highway has an orientation (e.g., east-west) and a vehicle traveling east has a sense. Knowing both the orientation of a line and the sense on the line gives direction. Changing a vector's orientation or sense changes its direction.

Example of a vector: Consider the traffic report "the vehicle is heading East at 5 $\frac{\text{m}}{\text{s}}$". It is ***convenient*** to name these two pieces of information (speed and direction) as a "***velocity vector***" and represent them mathematically as $5 * \overrightarrow{\text{East}}$ (direction is identified with an arrow and/or bold-face font or with a hat for a ***unit vector*** such as $\widehat{\mathbf{East}}$). The vehicle's speed is always a real non-negative number, equal to the ***magnitude*** of the velocity vector. The combination of ***magnitude*** and ***direction*** is a ***vector***.

For example, the vector $\vec{\mathbf{v}}$ describing a vehicle traveling with speed 5 to the East is graphically depicted to the right, and is written



$$\vec{\mathbf{v}} = 5 * \widehat{\mathbf{East}} \qquad \text{or} \qquad \vec{\mathbf{v}} = 5\,\widehat{\mathbf{East}}$$

A vehicle traveling with speed 5 to the West is

$$\vec{\mathbf{v}} = 5\,\widehat{\mathbf{West}} \qquad \text{or} \qquad \vec{\mathbf{v}} = \text{-}5\,\widehat{\mathbf{East}}$$

Note: The negative sign in $\text{-}5\,\widehat{\mathbf{East}}$ is associated with the vector's direction (the vector's magnitude is inherently non-negative). When a vector is written in terms of a scalar x that can be **positive** or zero or **negative**, e.g., as $x\,\widehat{\mathbf{East}}$, x is called the $\widehat{\mathbf{East}}$ ***measure*** of the vector, whereas the vector's non-negative ***magnitude*** is abs(x).

2.3 Zero vector $\vec{\mathbf{0}}$ and its properties

A ***zero vector*** $\vec{\mathbf{0}}$ is defined as a vector whose magnitude is zero.[2]

Addition of a vector $\vec{\mathbf{v}}$ with a zero vector:	$\vec{\mathbf{v}} + \vec{\mathbf{0}} = \vec{\mathbf{v}}$	
Dot product with a zero vector:	$\vec{\mathbf{v}} \cdot \vec{\mathbf{0}} \underset{(2)}{=} 0$	$\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors
Cross product with a zero vector:	$\vec{\mathbf{v}} \times \vec{\mathbf{0}} \underset{(5)}{=} \vec{\mathbf{0}}$	$\vec{\mathbf{0}}$ is ***parallel*** to all vectors
Derivative of the zero vector:	$\frac{{}^F d\vec{\mathbf{0}}}{dt} = \vec{\mathbf{0}}$	F is any reference frame

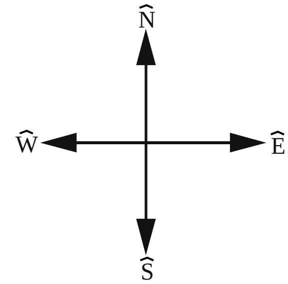
Vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are said to be "***perpendicular***" if $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$ whereas $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" if $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$.
Note: Some say $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" only if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have the same direction and anti-parallel if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have opposite directions.

[2] The direction of a zero vector $\vec{\mathbf{0}}$ is arbitrary and may be regarded as having **any** direction so that $\vec{\mathbf{0}}$ is ***parallel*** to all vectors, $\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors, all zero vectors are equal, and one may use the definite pronoun "the" instead of the indefinite "a" e.g., "the zero vector". It is improper to say the ***zero vector*** has no direction as a vector is **defined** to have both magnitude and direction. It is also improper to say a ***zero vector*** has all directions as a vector is defined to have a magnitude and **a** direction (as contrasted with a dyad which has 2 directions or triad which has 3 directions).

2.4 Unit $\widehat{\text{v}}$ectors

A ***unit vector*** is defined as a vector whose magnitude is 1, and is designated with a special hat, e.g., $\widehat{\mathbf{u}}$.

Unit vectors can be "***sign posts***", e.g., unit vectors $\widehat{\mathbf{N}}$, $\widehat{\mathbf{S}}$, $\widehat{\mathbf{W}}$, $\widehat{\mathbf{E}}$ associated with local Earth directions **North**, **South**, **West**, **East**, respectively. The direction of unit vectors are chosen to simplify communication and to produce efficient equations. Other useful "sign posts" are:



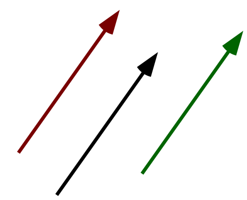
- Unit vector directed from one point to another point
- Unit vector directed locally vertical
- Unit vector parallel to the edge of an object
- Unit vector tangent to a curve or perpendicular to a surface

A unit vector can be defined so it has the same direction as an arbitrary non-zero vector $\vec{\mathbf{v}}$ by dividing $\vec{\mathbf{v}}$ by $|\vec{\mathbf{v}}|$ (the magnitude of $\vec{\mathbf{v}}$).
To avoid divide-by-zero problems during numerical computation, approximate the unit vector with a "small" positive real number ϵ in the denominator.

$$\boxed{\begin{aligned} \textbf{unit}\vec{\textbf{V}}\textbf{ector} &= \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \\ &\approx \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}| + \epsilon} \end{aligned}} \tag{1}$$

2.5 Equal vectors (=)

Two vectors are "equal" when they have the same magnitude and same direction.[a] Shown to the right are three ***equal vectors***. Although each has a different location, the vectors are equal because they have the same magnitude and direction.[b]

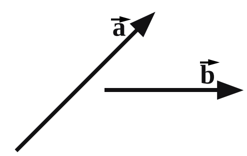


[a] Homework 2.6 draws vectors of different magnitude, ***orientation***, and ***sense***.

[b] Some vectors have additional properties. For example, a position vector is associated with two points. Two position vectors are ***equal position vectors*** when, they have the same magnitude, same direction, and are associated with the same points. Two force vectors are ***equal force vectors*** when they have the same magnitude, direction, and point of application.

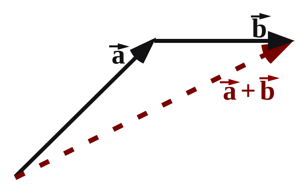
2.6 Vector addition (+)

As graphically shown to the right, adding two vectors $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ produces a vector.[a] First, vector $\vec{\mathbf{b}}$ is translated[b] so its tail is at the tip of $\vec{\mathbf{a}}$. Next, the vector $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ is drawn from the tail of $\vec{\mathbf{a}}$ to the tip of the translated $\vec{\mathbf{b}}$.



Properties of vector addition

Commutative law:	$\vec{\mathbf{a}} + \vec{\mathbf{b}} \;=\; \vec{\mathbf{b}} + \vec{\mathbf{a}}$
Associative law:	$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} \;=\; \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) \;=\; \vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}$
Addition of zero vector:	$\vec{\mathbf{a}} + \vec{\mathbf{0}} \;=\; \vec{\mathbf{a}}$



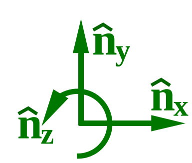
[a] It does not make sense to add vectors with different units, e.g., it is nonsensical to add a velocity vector with units of $\frac{\text{m}}{\text{s}}$ with an angular velocity vector with units of $\frac{\text{rad}}{\text{sec}}$.

[b] Translating $\vec{\mathbf{b}}$ does *not* change the magnitude or direction of $\vec{\mathbf{b}}$, and so produces an equal $\vec{\mathbf{b}}$.

Example: Vector addition (+)

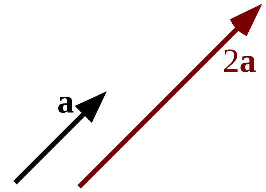
Shown to the right is an example of how to add vector $\vec{\mathbf{w}}$ to vector $\vec{\mathbf{v}}$, each which is expressed in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_\text{x}$, $\widehat{\mathbf{n}}_\text{y}$, $\widehat{\mathbf{n}}_\text{z}$.

$$\begin{array}{rcrcrcr} \vec{\mathbf{v}} & = & 7\,\widehat{\mathbf{n}}_\text{x} & + & 5\,\widehat{\mathbf{n}}_\text{y} & + & 4\,\widehat{\mathbf{n}}_\text{z} \\ +\;\vec{\mathbf{w}} & = & 2\,\widehat{\mathbf{n}}_\text{x} & + & 3\,\widehat{\mathbf{n}}_\text{y} & + & 2\,\widehat{\mathbf{n}}_\text{z} \\ \hline & & 9\,\widehat{\mathbf{n}}_\text{x} & + & 8\,\widehat{\mathbf{n}}_\text{y} & + & 6\,\widehat{\mathbf{n}}_\text{z} \end{array}$$

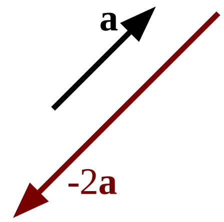


2.7 Vector multiplied or divided by a scalar $(*$ or $/)$

To the right is a graphical representation of multiplying a vector $\vec{\mathbf{a}}$ by a scalar.[a]



- Multiplying a vector by a **positive** number (other than 1) changes the vector's magnitude.
- Multiplying a vector by a **negative** number changes the vector's magnitude **and** reverses the ***sense*** of the vector.
- Dividing a vector $\vec{\mathbf{a}}$ by a scalar s_1 is defined as $\dfrac{\vec{\mathbf{a}}}{s_1} \triangleq \dfrac{1}{s_1} * \vec{\mathbf{a}}$



Properties of multiplication of a vector by a scalar s_1 or s_2

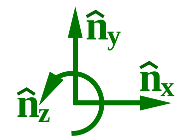
Commutative law:	$s_1 \vec{\mathbf{a}} = \vec{\mathbf{a}} s_1$
Associative law:	$s_1 (s_2 \vec{\mathbf{a}}) = (s_1 s_2) \vec{\mathbf{a}} = s_2 (s_1 \vec{\mathbf{a}}) = s_1 s_2 \vec{\mathbf{a}}$
Distributive law:	$(s_1 + s_2) \vec{\mathbf{a}} = s_1 \vec{\mathbf{a}} + s_2 \vec{\mathbf{a}}$
Distributive law:	$s_1 (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = s_1 \vec{\mathbf{a}} + s_1 \vec{\mathbf{b}}$
Multiplication by zero:	$0 * \vec{\mathbf{a}} = \vec{\mathbf{0}}$

[a] Homework 2.9 multiplies a vector $\vec{\mathbf{b}}$ by various scalars.

Example: Vector scalar multiplication and division $(*$ and $/)$

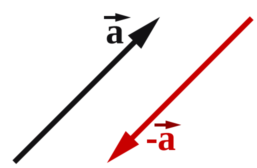
Given: $\vec{\mathbf{v}} = 7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 4\,\hat{\mathbf{n}}_z \quad \Rightarrow$

$$5\,\vec{\mathbf{v}} = 35\,\hat{\mathbf{n}}_x + 25\,\hat{\mathbf{n}}_y + 20\,\hat{\mathbf{n}}_z$$
$$\frac{\vec{\mathbf{v}}}{-2} = -3.5\,\hat{\mathbf{n}}_x - 2.5\,\hat{\mathbf{n}}_y - 2\,\hat{\mathbf{n}}_z$$



2.8 Vector negation and subtraction $(-)$

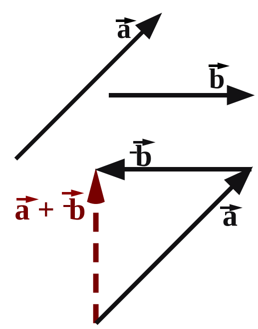
Negation: A graphical representation of negating a vector $\vec{\mathbf{a}}$ is shown right. Negating a vector (multiplying the vector by -1) changes the ***sense*** of a vector without changing its magnitude or orientation. In other words, multiplying a vector by -1 reverses the sense of the vector (it points in the opposite direction).



Subtraction: As drawn right, the process of subtracting a vector $\vec{\mathbf{b}}$ from a vector $\vec{\mathbf{a}}$ is simply addition and negation.[a]

$$\boxed{\vec{\mathbf{a}} - \vec{\mathbf{b}} \triangleq \vec{\mathbf{a}} + -\vec{\mathbf{b}}}$$

After negating vector $\vec{\mathbf{b}}$, it is translated so the tail of $-\vec{\mathbf{b}}$ is at the tip of $\vec{\mathbf{a}}$. Next, vector $\vec{\mathbf{a}} + -\vec{\mathbf{b}}$ is drawn from the tail of $\vec{\mathbf{a}}$ to the tip of the translated $-\vec{\mathbf{b}}$.

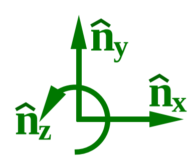


[a] In most (or all) mathematical processes, subtraction is defined as negation and addition.

Example: Vector subtraction $(-)$

Shown right is an example of how to subtract vector $\vec{\mathbf{w}}$ from vector $\vec{\mathbf{v}}$, each which is expressed in terms of orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$.

$$\begin{array}{rcl} \vec{\mathbf{v}} & = & 7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 4\,\hat{\mathbf{n}}_z \\ -\ \vec{\mathbf{w}} & = & 2\,\hat{\mathbf{n}}_x + 3\,\hat{\mathbf{n}}_y + 2\,\hat{\mathbf{n}}_z \\ \hline & & 5\,\hat{\mathbf{n}}_x + 2\,\hat{\mathbf{n}}_y + 2\,\hat{\mathbf{n}}_z \end{array}$$



2.9 Vector dot product $(\cdot)$

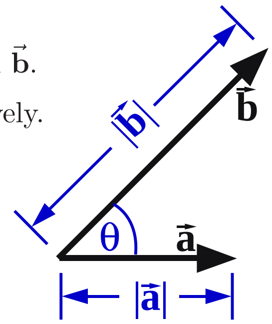
Equation (2) defines the ***dot product*** of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

- $|\vec{\mathbf{a}}|$ and $|\vec{\mathbf{b}}|$ are the magnitudes of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, respectively.
- θ is the smallest angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $(0 \le \theta \le \pi)$.

Equation (3) is a rearrangement of equation (2) that is useful for calculating the angle θ between two vectors.

Note: $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***perpendicular***" when $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.

Note: Dot-products encapsulate the ***law of cosines***.



$$\boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \triangleq |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\cos(\theta)} \qquad (2)$$

$$\boxed{\cos(\theta) \underset{(2)}{=} \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|\;|\vec{\mathbf{b}}|}} \qquad (3)$$

Use **acos** to calculate θ.

Equation (2) shows $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{v}}|^2$. Hence, the dot product can calculate a vector's ***magnitude*** as shown for $|\vec{\mathbf{v}}|$ in equation (4).

Equation (4) also defines ***vector exponentiation*** $\vec{\mathbf{v}}^n$ (vector $\vec{\mathbf{v}}$ raised to scalar power n) as a non-negative scalar.

Example: Kinetic energy $K = \frac{1}{2}\, m\, \vec{\mathbf{v}}^2 \underset{(4)}{=} \frac{1}{2}\, m\, \vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$

$$\boxed{\begin{aligned} \vec{\mathbf{v}}^2 \triangleq |\vec{\mathbf{v}}|^2 &= \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \\ |\vec{\mathbf{v}}| &= +\sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}} \\ \vec{\mathbf{v}}^n \triangleq |\vec{\mathbf{v}}|^n &= +(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})^{\frac{n}{2}} \end{aligned}} \qquad (4)$$

2.9.1 Properties of the dot-product $(\cdot)$

Dot product with a zero vector	$\vec{\mathbf{a}} \cdot \vec{\mathbf{0}} = 0$	
Dot product of ***perpendicular*** vectors	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$	if $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$
Dot product of parallel vectors	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \pm\,|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|$	if $\vec{\mathbf{a}} \parallel \vec{\mathbf{b}}$
Dot product with vectors scaled by s_1 and s_2	$s_1\vec{\mathbf{a}} \cdot s_2\vec{\mathbf{b}} = s_1\, s_2\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$	
Commutative law	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$	
Distributive law	$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$	
Distributive law	$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot (\vec{\mathbf{c}} + \vec{\mathbf{d}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{d}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{d}}$	

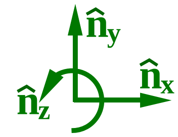
Note: The distributive law for dot-products and cross-products is proved in [32, pgs. 23-24, 32-34].

2.9.2 Uses for the dot-product $(\cdot)$

- Calculating an ***angle*** between two vectors [see equation (3) and example in Section 3.3] or determining when two vectors are ***perpendicular***, e.g., $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.
- Calculating a vector's ***magnitude*** [see equation (4) and ***distance*** examples in Sections 3.2 and 3.3].
- Changing a ***vector equation*** into a ***scalar equation*** (see Homework 2.31).
- Calculating a ***unit vector*** in the direction of a vector $\vec{\mathbf{v}}$ [see equation (1)] $\boxed{\textbf{unit}\vec{\textbf{V}}\textbf{ector} \underset{(1)}{=} \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}}$
- ***Projection*** of a vector $\vec{\mathbf{v}}$ in the direction of $\vec{\mathbf{b}}$ is defined: $\boxed{\vec{\mathbf{v}} \cdot \frac{\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|}}$

  See Section 4.2 for ***projections***, ***measures***, ***coefficients***, ***components***.

2.9.3 Special case: Dot-products with orthogonal unit vectors

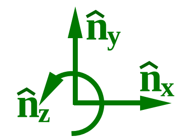


When $\widehat{\mathbf{n}}_\mathrm{x}$, $\widehat{\mathbf{n}}_\mathrm{y}$, $\widehat{\mathbf{n}}_\mathrm{z}$ are ***orthogonal unit*** vectors, it can be shown (see Homework 2.4)

$$(a_x\,\widehat{\mathbf{n}}_\mathrm{x} + a_y\,\widehat{\mathbf{n}}_\mathrm{y} + a_z\,\widehat{\mathbf{n}}_\mathrm{z}) \cdot (b_x\,\widehat{\mathbf{n}}_\mathrm{x} + b_y\,\widehat{\mathbf{n}}_\mathrm{y} + b_z\,\widehat{\mathbf{n}}_\mathrm{z}) \;=\; a_x\,b_x + a_y\,b_y + a_z\,b_z$$

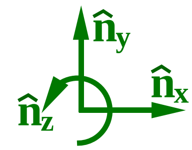
Optional: Special case of dot product as matrix multiplication

When one defines $\vec{\mathbf{a}} \triangleq a_x\,\widehat{\mathbf{n}}_\mathrm{x} + a_y\,\widehat{\mathbf{n}}_\mathrm{y} + a_z\,\widehat{\mathbf{n}}_\mathrm{z}$ and $\vec{\mathbf{b}} \triangleq b_x\,\widehat{\mathbf{n}}_\mathrm{x} + b_y\,\widehat{\mathbf{n}}_\mathrm{y} + b_z\,\widehat{\mathbf{n}}_\mathrm{z}$ in terms of the ***orthogonal unit*** vectors $\widehat{\mathbf{n}}_\mathrm{x}$, $\widehat{\mathbf{n}}_\mathrm{y}$, $\widehat{\mathbf{n}}_\mathrm{z}$, the dot-product $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ is related to the multiplication of the $\widehat{\mathbf{n}}_\mathrm{x}$, $\widehat{\mathbf{n}}_\mathrm{y}$, $\widehat{\mathbf{n}}_\mathrm{z}$ row matrix representation of $\vec{\mathbf{a}}$ with the $\widehat{\mathbf{n}}_\mathrm{x}$, $\widehat{\mathbf{n}}_\mathrm{y}$, $\widehat{\mathbf{n}}_\mathrm{z}$ column matrix representation of $\vec{\mathbf{b}}$ as



$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \;=\; \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}_{\widehat{\mathbf{n}}_\mathrm{xyz}} * \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}_{\widehat{\mathbf{n}}_\mathrm{xyz}} \;=\; \begin{bmatrix} a_x\,b_x + a_y\,b_y + a_z\,b_z \end{bmatrix}$$

2.9.4 Examples: Vector dot-products $(\cdot)$



The following shows how to use dot-products with the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, each which is expressed in terms of the orthogonal unit vectors $\widehat{\mathbf{n}}_\mathrm{x}$, $\widehat{\mathbf{n}}_\mathrm{y}$, $\widehat{\mathbf{n}}_\mathrm{z}$ shown to the right.

$$\vec{\mathbf{v}} \;=\; 7\,\widehat{\mathbf{n}}_\mathrm{x} + 5\,\widehat{\mathbf{n}}_\mathrm{y} + 4\,\widehat{\mathbf{n}}_\mathrm{z}\vec{\mathbf{w}} \;=\; 2\,\widehat{\mathbf{n}}_\mathrm{x} + 3\,\widehat{\mathbf{n}}_\mathrm{y} + 2\,\widehat{\mathbf{n}}_\mathrm{z}$$

$\widehat{\mathbf{n}}_\mathrm{x}$ measure of $\vec{\mathbf{v}}$ $\qquad$ $\vec{\mathbf{v}} \cdot \widehat{\mathbf{n}}_\mathrm{x} \;=\; 7$ (**measures** how much of $\vec{\mathbf{v}}$ is in the $\widehat{\mathbf{n}}_\mathrm{x}$ direction)

$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \;=\; 7^2 + 5^2 + 4^2 \;=\; 90$ $\qquad$ $|\vec{\mathbf{v}}| \;=\; \sqrt{90} \;\approx\; 9.4868$

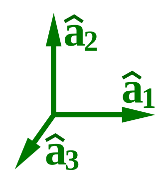
$\vec{\mathbf{w}} \cdot \vec{\mathbf{w}} \;=\; 2^2 + 3^2 + 2^2 \;=\; 17$ $\qquad$ $|\vec{\mathbf{w}}| \;=\; \sqrt{17} \;\approx\; 4.1231$

Unit vector in the direction of $\vec{\mathbf{v}}$: $\qquad \dfrac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \;=\; \dfrac{7\,\widehat{\mathbf{n}}_\mathrm{x} + 5\,\widehat{\mathbf{n}}_\mathrm{y} + 4\,\widehat{\mathbf{n}}_\mathrm{z}}{\sqrt{90}} \;\approx\; 0.738\,\widehat{\mathbf{n}}_\mathrm{x} + 0.527\,\widehat{\mathbf{n}}_\mathrm{y} + 0.422\,\widehat{\mathbf{n}}_\mathrm{z}$

Unit vector in the direction of $\vec{\mathbf{w}}$: $\qquad \dfrac{\vec{\mathbf{w}}}{|\vec{\mathbf{w}}|} \;=\; \dfrac{2\,\widehat{\mathbf{n}}_\mathrm{x} + 3\,\widehat{\mathbf{n}}_\mathrm{y} + 2\,\widehat{\mathbf{n}}_\mathrm{z}}{\sqrt{17}} \;\approx\; 0.485\,\widehat{\mathbf{n}}_\mathrm{x} + 0.728\,\widehat{\mathbf{n}}_\mathrm{y} + 0.485\,\widehat{\mathbf{n}}_\mathrm{z}$

$\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} \;=\; 7*2 + 5*3 + 4*2 \;=\; 37 \qquad \angle(\vec{\mathbf{v}}, \vec{\mathbf{w}}) \;=\; \mathrm{acos}\big(\dfrac{37}{\sqrt{90}\,\sqrt{17}}\big) \;\approx\; 0.33 \text{ rad} \;\approx\; 18.93^\circ$

2.9.5 Dot-products to change vector equations to scalar equations (see Hw 1.31)



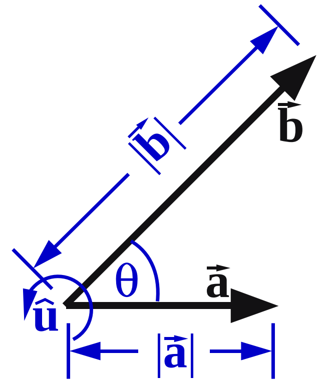
One way to form up to three linearly independent scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ is by dot-multiplying $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ with three orthogonal unit vectors $\widehat{\mathbf{a}}_1$, $\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_3$, i.e.,

$$\text{if} \quad \vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \boxed{\vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_1 = 0 \qquad \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_2 = 0 \qquad \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_3 = 0}$$

Section 2.11.2 describes another way to form three ***different*** scalar equations from $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Courtesy Accuray Inc.. Dot-products are heavily used in radiation and other medical equipment.

2.10 Vector cross product $(\times)$



The ***cross product*** of a vector $\vec{\mathbf{a}}$ with a vector $\vec{\mathbf{b}}$ is defined in equation (5).

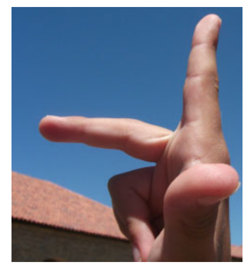
- $|\vec{\mathbf{a}}|$ and $|\vec{\mathbf{b}}|$ are the magnitudes of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, respectively
- θ is the smallest angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $(0 \le \theta \le \pi)$.
- $\hat{\mathbf{u}}$ is the unit vector **perpendicular** to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.
  The direction of $\hat{\mathbf{u}}$ is determined by the ***right-hand rule***.[a]

  Note: $|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\sin(\theta)$ [the coefficient of $\hat{\mathbf{u}}$ in equation (5)] is inherently ***non-negative*** because $\sin(\theta) \ge 0$ since $0 \le \theta \le \pi$. Hence, $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\sin(\theta)$.

[a] The right-hand rule is a convention, much like driving on the right-hand side of the road in North America. Until 1965, the Soviet Union used the left-hand rule.

$$\boxed{\vec{\mathbf{a}} \times \vec{\mathbf{b}} \triangleq |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\sin(\theta)\,\hat{\mathbf{u}}} \qquad (5)$$

2.10.1 Properties of the cross-product $(\times)$



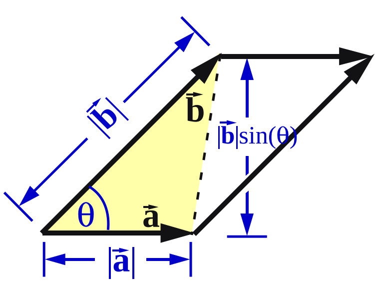
Cross product with a zero vector	$\vec{\mathbf{a}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}}$
Cross product of a vector with itself	$\vec{\mathbf{a}} \times \vec{\mathbf{a}} = \vec{\mathbf{0}}$
Cross product of ***parallel*** vectors	$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$ if $\vec{\mathbf{a}} \parallel \vec{\mathbf{b}}$
Cross product with vectors scaled by s_1 and s_2	$s_1\vec{\mathbf{a}} \times s_2\vec{\mathbf{b}} = s_1\,s_2\,(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$
Cross products are ***not*** commutative	$\boxed{\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}}$ (6)
Distributive law	$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$
Distributive law	$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \times (\vec{\mathbf{c}} + \vec{\mathbf{d}}) = \vec{\mathbf{a}} \times \vec{\mathbf{c}} + \vec{\mathbf{a}} \times \vec{\mathbf{d}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times \vec{\mathbf{d}}$
Cross products are ***not*** associative	$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) \ne (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}}$
Vector triple cross product	$\boxed{\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})}$ (7)

A mnemonic for $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$ is "**bac**k **cab**" - as in were you born in the back of a cab?
Many proofs of this formula resolve $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ into orthogonal unit vectors (e.g., $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$) and equate components.

2.10.2 Uses for the cross-product $(\times)$

Several uses for the cross-product in geometry, statics, and motion analysis, include calculating:

- ***Perpendicular*** vectors, e.g., $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$
- ***Moment*** of a force or translational momentum, e.g., $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$ and $\vec{\mathbf{r}} \times m\,\vec{\mathbf{v}}$
- ***Velocity/acceleration*** formulas, e.g., $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$ and $\vec{\mathbf{a}} = \vec{\boldsymbol{\alpha}} \times \vec{\mathbf{r}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$
- ***Area of a triangle*** whose sides have length $|\vec{\mathbf{a}}|$ and $|\vec{\mathbf{b}}|$

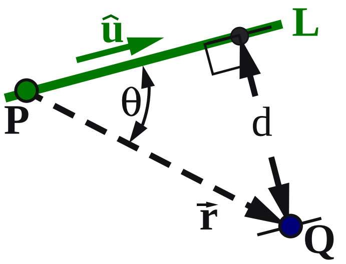


The ***area of a triangle*** Δ is half the area of a parallelogram.[a] [b]
A geometrical interpretation of $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$ is the ***area of a parallelogram*** having sides of length $|\vec{\mathbf{a}}|$ and $|\vec{\mathbf{b}}|$, hence

$$\boxed{\Delta(\vec{\mathbf{a}},\, \vec{\mathbf{b}}) = \tfrac{1}{2}\,|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|} \qquad (8)$$

[a] Homework 2.14 shows the utility of equation (8) for ***surveying***.
[b] Section 3.3 shows the utility of a cross-product for area calculations.

- ***Distance*** d between a line L and a point Q.

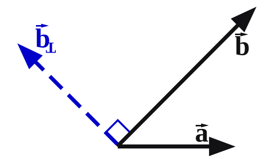


The line L (shown left) passes through point P and is parallel to the unit vector $\hat{\mathbf{u}}$. The ***distance*** d between line L and a point Q can be calculated as

$$\boxed{\mathrm{d} \;=\; \left|\vec{\mathbf{r}}^{\,Q/P} \times \hat{\mathbf{u}}\right| \;\underset{(5)}{=}\; \left|\vec{\mathbf{r}}^{\,Q/P}\right| \sin(\theta)} \tag{9}$$

Note: See example in Hw 1.26. Other distance calculations are in Sections 3.2 and 3.3.

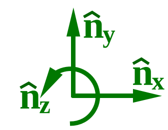
- The vector $\vec{\mathbf{b}}_\perp$ (shown right) is perpendicular to $\vec{\mathbf{b}}$ and is in the plane containing both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. It is calculated with the ***vector triple cross product***:



$$\vec{\mathbf{b}}_\perp \;=\; \left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right) \times \vec{\mathbf{b}}$$

In general, $|\vec{\mathbf{b}}_\perp| \neq |\vec{\mathbf{b}}|$ and $\vec{\mathbf{b}}_\perp$ is not perpendicular to $\vec{\mathbf{a}}$.

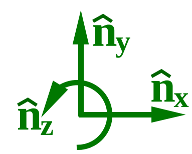
2.10.3 Special case: Cross-products with right-handed, orthogonal, unit vectors



When $\hat{\mathbf{n}}_\mathrm{x}$, $\hat{\mathbf{n}}_\mathrm{y}$, $\hat{\mathbf{n}}_\mathrm{z}$ are ***orthogonal unit*** vectors, it can be shown (see Homework 2.13) that the cross product of $\vec{\mathbf{a}} = a_x\,\hat{\mathbf{n}}_\mathrm{x} + a_y\,\hat{\mathbf{n}}_\mathrm{y} + a_z\,\hat{\mathbf{n}}_\mathrm{z}$ with $\vec{\mathbf{b}} = b_x\,\hat{\mathbf{n}}_\mathrm{x} + b_y\,\hat{\mathbf{n}}_\mathrm{y} + b_z\,\hat{\mathbf{n}}_\mathrm{z}$ happens to be equal to the ***determinant*** of the following matrix.

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \;=\; \det\begin{bmatrix} \hat{\mathbf{n}}_\mathrm{x} & \hat{\mathbf{n}}_\mathrm{y} & \hat{\mathbf{n}}_\mathrm{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \;=\; (a_y\,b_z - a_z\,b_y)\,\hat{\mathbf{n}}_\mathrm{x} \;-\; (a_x\,b_z - a_z\,b_x)\,\hat{\mathbf{n}}_\mathrm{y} \;+\; (a_x\,b_y - a_y\,b_x)\,\hat{\mathbf{n}}_\mathrm{z}$$

2.10.4 Examples: Vector cross-products ($\times$)



The following shows how to use cross-products with the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, each which is expressed in terms of the orthogonal unit vectors $\hat{\mathbf{n}}_\mathrm{x}$, $\hat{\mathbf{n}}_\mathrm{y}$, $\hat{\mathbf{n}}_\mathrm{z}$ shown to the right.

$$\vec{\mathbf{v}} \;=\; 7\,\hat{\mathbf{n}}_\mathrm{x} + 5\,\hat{\mathbf{n}}_\mathrm{y} + 4\,\hat{\mathbf{n}}_\mathrm{z}$$

$$\vec{\mathbf{w}} \;=\; 2\,\hat{\mathbf{n}}_\mathrm{x} + 3\,\hat{\mathbf{n}}_\mathrm{y} + 2\,\hat{\mathbf{n}}_\mathrm{z}$$

$$\vec{\mathbf{v}} \times \vec{\mathbf{w}} \;=\; \det\begin{bmatrix} \hat{\mathbf{n}}_\mathrm{x} & \hat{\mathbf{n}}_\mathrm{y} & \hat{\mathbf{n}}_\mathrm{z} \\ 7 & 5 & 4 \\ 2 & 3 & 2 \end{bmatrix} \;=\; \text{-}2\,\hat{\mathbf{n}}_\mathrm{x} - 6\,\hat{\mathbf{n}}_\mathrm{y} + 11\,\hat{\mathbf{n}}_\mathrm{z}$$

Area from vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$: $\quad \Delta(\vec{\mathbf{v}},\,\vec{\mathbf{w}}) \;=\; \frac{1}{2}\,|\vec{\mathbf{v}} \times \vec{\mathbf{w}}| \;=\; \frac{1}{2}\sqrt{\text{-}2^2 + \text{-}6^2 + 11^2} \;=\; \frac{\sqrt{161}}{2} \;\approx\; 6.344$

$$\vec{\mathbf{v}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) \;=\; \det\begin{bmatrix} \hat{\mathbf{n}}_\mathrm{x} & \hat{\mathbf{n}}_\mathrm{y} & \hat{\mathbf{n}}_\mathrm{z} \\ 7 & 5 & 4 \\ \text{-}2 & \text{-}6 & 11 \end{bmatrix} \;=\; 79\,\hat{\mathbf{n}}_\mathrm{x} - 85\,\hat{\mathbf{n}}_\mathrm{y} - 32\,\hat{\mathbf{n}}_\mathrm{z}$$

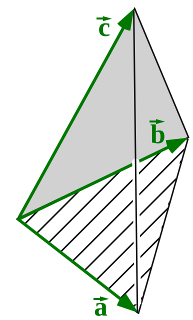
2.11 Optional: Scalar triple product $(\cdot \; \times \;\text{ or }\; \times \; \cdot)$

The ***scalar triple product*** of vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ is the scalar defined in the various ways shown in equation (10).

$$\text{ScalarTripleProduct} \triangleq \boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} \;=\; \vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}} \;=\; \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} \times \vec{\mathbf{a}} \;=\; \vec{\mathbf{b}} \times \vec{\mathbf{c}} \cdot \vec{\mathbf{a}} \qquad (10)$$

Although parentheses make equation (10) clearer, i.e., ScalarTripleProduct $\triangleq \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$, the parentheses are unnecessary because the cross product $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ **must** be performed before the dot product for a sensible result to be produced.

2.11.1 Scalar triple product and the volume of a tetrahedron

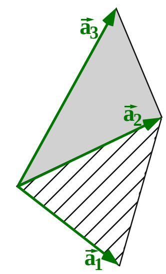


A geometrical interpretation of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$ is the ***volume of a parallelepiped*** having sides of length $|\vec{\mathbf{a}}|$, $|\vec{\mathbf{b}}|$, and $|\vec{\mathbf{c}}|$. The formula for the ***volume of a tetrahedron*** whose sides are described by the vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ is

$$\text{Tetrahedron Volume} \;=\; \frac{1}{6}\; \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$$

This formula is used for volume calculations (e.g., highway ***surveying*** cut and fill), 3D ***CAD***, solid geometry, and mass property calculations.

2.11.2 $(\times \;\cdot)$ to change vector equations to scalar equations (see Hw 1.31)



Section 2.9.5 showed one method to form scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$. A 2^{nd} method expresses $\vec{\mathbf{v}}$ in terms of three non-coplanar (but not necessarily orthogonal or unit) vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3$, and writes the equally valid (but generally different) set of linearly independent scalar equations shown below.

$$\text{Method 2:}\quad \text{if}\quad \vec{\mathbf{v}} \;=\; v_1\, \vec{\mathbf{a}}_1 \;+\; v_2\, \vec{\mathbf{a}}_2 \;+\; v_3\, \vec{\mathbf{a}}_3 \;=\; \vec{\mathbf{0}} \quad\Rightarrow\quad \boxed{v_1 = 0 \qquad v_2 = 0 \qquad v_3 = 0}$$

Note: The proof that $v_i = 0$ $(i = 1, 2, 3)$ follows directly by substituting $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ into equation (4.2).

Vectors are used with ***surveying data*** for volume cut-and-fill dirt calculations for highway construction

