Chapter 2

Vectors

Summary  (see examples in Hw 1, 2, 3)

Circa 1900 A.D., J. Williard Gibbs invented a useful combination of magnitude and direction called vectors and their higher-dimensional counterparts dyadics, triadics, and polyadics. Vectors are an important geometrical tool e.g., for surveying, motion analysis, lasers, optics, computer graphics, animation, CAD/CAE (computer aided drawing/engineering), and FEA (finite element analysis).

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2.1 Examples of scalars, vectors, and dyadics

- A **scalar** is a non-directional quantity (e.g., a real number). Examples include:

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<th>time</th>
<th>density</th>
<th>volume</th>
<th>mass</th>
<th>moment of inertia</th>
<th>temperature</th>
<th>distance</th>
<th>speed</th>
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- A **vector** is a quantity that has magnitude and **one** associated direction. For example, a **velocity vector** has speed (how fast something is moving) and direction (which way it is going). A **force vector** has magnitude (how hard something is pushed) and direction (which way it is shoved). Examples include:

<table>
<thead>
<tr>
<th>position vector</th>
<th>velocity</th>
<th>acceleration</th>
<th>linear momentum</th>
<th>force</th>
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<tr>
<td>impulse</td>
<td>angular velocity</td>
<td>angular acceleration</td>
<td>angular momentum</td>
<td>torque</td>
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- A **dyad** is a quantity with magnitude and **two** associated directions. For example, **stress** associates with area and force (both regarded as vectors). A **dyadic** is the **sum of dyads**. For example, an **inertia dyadic** (Chapter 14) is the sum of dyads associated with moments and products of inertia.
2.2 Definition of a vector

A **vector** is defined as a quantity having **magnitude** and **direction**. Vectors are represented pictorially with straight or curved arrows (examples below). Vectors are typeset with an arrow and bold-faced font, e.g., \( \vec{v} \) denotes a vector.

Certain vectors have additional special properties. For example, a **position vector** \( \vec{r} \) is associated with two points and has units of distance.

**Example of a vector:** Consider the statement “the car is moving East at 5 m/s.” It is convenient to represent the car’s speed and direction with the velocity vector \( \vec{v} = 5 \hat{\text{East}} \) (a hat designates the direction East as a unit vector).

The velocity of a car moving West with speed 5 is \( \vec{v} = 5 \hat{\text{West}} = -5 \hat{\text{East}} \). The negative sign in \(-5 \hat{\text{East}}\) is associated with the vector’s direction (the vector’s magnitude is inherently non-negative). When a vector is written in terms of a scalar \( x \) that can be positive or zero or negative, e.g., as \( x \hat{\text{East}} \), \( x \) is called the \( \hat{\text{East}} \) measure of the vector, whereas the vector’s non-negative magnitude is \( \text{abs}(x) \).

2.3 Zero vector \( \vec{0} \) and its properties

A **zero vector** \( \vec{0} \) is defined as a vector whose magnitude is zero.

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<th>Statement</th>
<th>Equation</th>
<th>Note</th>
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<td>Addition of a vector ( \vec{v} ) with a zero vector:</td>
<td>( \vec{v} + \vec{0} = \vec{v} )</td>
<td>( \vec{0} ) is <strong>perpendicular</strong> to all vectors</td>
</tr>
<tr>
<td>Dot product with a zero vector:</td>
<td>( \vec{v} \cdot \vec{0} = 0 )</td>
<td>(2)</td>
</tr>
<tr>
<td>Cross product with a zero vector:</td>
<td>( \vec{v} \times \vec{0} = \vec{0} )</td>
<td>( \vec{0} ) is <strong>parallel</strong> to all vectors</td>
</tr>
<tr>
<td>Derivative of the zero vector:</td>
<td>( \frac{d\vec{0}}{dt} = \vec{0} )</td>
<td>( F ) is any reference frame</td>
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Vectors \( \vec{a} \) and \( \vec{b} \) are said to be “**perpendicular**” if \( \vec{a} \cdot \vec{b} = 0 \) whereas \( \vec{a} \) and \( \vec{b} \) are “**parallel**” if \( \vec{a} \times \vec{b} = \vec{0} \).

Note: Some say \( \vec{a} \) and \( \vec{b} \) are “**parallel**” only if \( \vec{a} \) and \( \vec{b} \) have the same direction and “anti-parallel” if \( \vec{a} \) and \( \vec{b} \) have opposite directions.

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1The direction of a zero vector \( \vec{0} \) is arbitrary and may be regarded as having **any** direction so that \( \vec{0} \) is **parallel** to all vectors, \( \vec{0} \) is **perpendicular** to all vectors, all zero vectors are equal, and one may use the definite pronoun “the” instead of the indefinite “a” e.g., “the zero vector”. It is improper to say the **zero vector** has no direction as a vector is **defined** to have both magnitude and direction. It is also improper to say a **zero vector** has all directions as a vector is defined to have a magnitude and a direction (as contrasted with a dyad which has 2 directions or triad which has 3 directions).
2.4 Unit Vectors

A unit vector is defined as a vector whose magnitude is 1, and is typeset with a special hat, e.g., $\hat{u}$. Unit vectors can be “sign posts”, e.g., unit vectors $\hat{N}, \hat{S}, \hat{W}, \hat{E}$ for local Earth directions North, South, West, East. The direction of unit vectors are chosen to simplify communication and to produce efficient equations. Other useful “sign posts” are:

- Unit vector directed from one point to another point
- Unit vector directed locally vertical
- Unit vector parallel to the edge of an object
- Unit vector tangent to a curve or perpendicular to a surface

A unit vector can be defined so it has the same direction as an arbitrary non-zero vector $\vec{v}$ by dividing $\vec{v}$ by $|\vec{v}|$ (the magnitude of $\vec{v}$).

$$\text{unitVector} = \frac{\vec{v}}{|\vec{v}|} \approx \frac{\vec{v}}{|\vec{v}| + \epsilon} \quad (1)$$

2.5 Equal vectors ( = )

Two vectors are “equal” when they have the same magnitude and same direction. Shown to the right are three equal vectors. Although each has a different location, the vectors are equal because they have the same magnitude and direction.$^b$

Some vectors have additional properties. For example, a position vector is associated with two points. Two position vectors are equal position vectors when, they have the same magnitude, same direction, and are associated with the same points. Two force vectors are equal force vectors when they have the same magnitude, direction, and point of application.

2.6 Vector addition (+)

As graphically shown to the right, adding two vectors $\vec{a} + \vec{b}$ produces a vector.$^a$

First, vector $\vec{b}$ is translated$^b$ so its tail is at the tip of $\vec{a}$. Next, the vector $\vec{a} + \vec{b}$ is drawn from the tail of $\vec{a}$ to the tip of the translated $\vec{b}$.

**Properties of vector addition**

- Commutative law: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- Associative law: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) = \vec{a} + \vec{b} + \vec{c}$
- Addition of zero vector: $\vec{a} + \vec{0} = \vec{a}$

$^a$It does not make sense to add vectors with different units, e.g., it is nonsensical to add a velocity vector with units of $m/s$ with an angular velocity vector with units of $rad/sec$.

$^b$Translating $\vec{b}$ does not change the magnitude or direction of $\vec{b}$, and so produces an equal $\vec{b}$.

**Example: Vector addition (+)**

Shown to the right is an example of how to add vector $\vec{w}$ to vector $\vec{v}$, each which is expressed in terms of orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$.

$$\vec{v} = 7\hat{n}_x + 5\hat{n}_y + 4\hat{n}_z$$

$$\vec{v} + \vec{w} = 2\hat{n}_x + 3\hat{n}_y + 2\hat{n}_z$$

$$\vec{v} + \vec{w} = 9\hat{n}_x + 8\hat{n}_y + 6\hat{n}_z$$
2.7 Vector multiplied or divided by a scalar (∗ or /)

To the right is a graphical representation of multiplying an arbitrary vector \( \vec{a} \) by real scalars.

- Multiplying a vector by a **positive** number (other than 1) changes the vector’s magnitude.
- Multiplying a vector by a **negative** number changes the vector’s magnitude and reverses the **sense** of the vector.
- Dividing a vector \( \vec{a} \) by a scalar \( s \) is defined as \( \frac{\vec{a}}{s} \triangleq \frac{1}{s} \ast \vec{a} \).

**Properties of multiplication of a vector by a scalar \( s \)**

- **Commutative law:** \( s_1 \ast \vec{a} = \vec{a} \ast s_1 \)
- **Associative law:** \( s_1 \ast (s_2 \ast \vec{a}) = (s_1 \ast s_2) \ast \vec{a} = s_1 \ast s_2 \ast \vec{a} \)
- **Distributive law:** \( (s_1 + s_2) \ast \vec{a} = s_1 \ast \vec{a} + s_2 \ast \vec{a} \)
- **Distributive law:** \( s_1 \ast (\vec{a} + \vec{b}) = s_1 \ast \vec{a} + s_1 \ast \vec{b} \)
- Multiplication by zero: \( 0 \ast \vec{a} = \vec{0} \)

**Example: Vector scalar multiplication and division (∗ and /)**

Given: \( \vec{v} = 7 \hat{n}_x + 5 \hat{n}_y + 4 \hat{n}_z \)
then: \( 5 \vec{v} = 35 \hat{n}_x + 25 \hat{n}_y + 20 \hat{n}_z \) and \( \frac{\vec{v}}{5} = -3.5 \hat{n}_x - 2.5 \hat{n}_y - 2 \hat{n}_z \)

2.8 Vector negation and subtraction (−)

**Negation:** As shown right, negating a vector (multiplying by \(-1\)) reverses the vector’s **sense** (it points in the opposite direction). Negation does not change the vector’s magnitude or orientation.

**Subtraction:** As the drawing to the right shows, subtracting a vector \( \vec{b} \) from a vector \( \vec{a} \) is simply addition and negation.\(^a\)

\[ \vec{a} - \vec{b} \triangleq \vec{a} + \vec{-b} \]

After negating vector \( \vec{b} \), it is translated so the tail of \( \vec{-b} \) is at the tip of \( \vec{a} \).

Next, vector \( \vec{a} + \vec{-b} \) is drawn from the tail of \( \vec{a} \) to the tip of the translated \( \vec{-b} \).

\(^a\)In most/all mathematics, subtraction is defined as negation and addition.

**Example: Vector subtraction (−)**

Shown right is an example of how to subtract vector \( \vec{w} \) from vector \( \vec{v} \), when each is expressed in terms of orthogonal unit vectors \( \hat{n}_x, \hat{n}_y, \hat{n}_z \).

\[
\begin{align*}
\vec{v} &= 7 \hat{n}_x + 5 \hat{n}_y + 4 \hat{n}_z \\
- \vec{w} &= 2 \hat{n}_x + 3 \hat{n}_y + 2 \hat{n}_z \\
\vec{v} - \vec{w} &= 5 \hat{n}_x + 2 \hat{n}_y + 2 \hat{n}_z
\end{align*}
\]
2.9 Vector dot product (·)

Equation (2) defines the dot product of vectors \( \mathbf{a} \) and \( \mathbf{b} \).
- \( |\mathbf{a}| \) and \( |\mathbf{b}| \) are the magnitudes of \( \mathbf{a} \) and \( \mathbf{b} \), respectively.
- \( \theta \) is the smallest angle between \( \mathbf{a} \) and \( \mathbf{b} \) (0 ≤ \( \theta \) ≤ π).

Equation (3) is a rearrangement of equation (2) that is useful for calculating the angle \( \theta \) between two vectors.

Note: \( \mathbf{a} \) and \( \mathbf{b} \) are “perpendicular” when \( \mathbf{a} \cdot \mathbf{b} = 0 \).

Equation (2) shows \( \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \). Hence, the dot product can calculate a vector’s magnitude as shown for |\( \mathbf{v} \) in equation (4).

Equation (4) also defines vector exponentiation \( \mathbf{v}^n \) (vector \( \mathbf{v} \) raised to scalar power \( n \)) as a non-negative scalar.

Example: Kinetic energy \( K = \frac{1}{2} m \mathbf{v}^2 = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} \).

2.9.1 Properties of the dot-product (·)

| Dot product with a zero vector | \( \mathbf{a} \cdot \mathbf{0} = 0 \) |
| Dot product of perpendicular vectors | \( \mathbf{a} \cdot \mathbf{b} = 0 \) if \( \mathbf{a} \parallel \mathbf{b} \) |
| Dot product of parallel vectors | \( \mathbf{a} \cdot \mathbf{b} = \pm |\mathbf{a}| |\mathbf{b}| \) if \( \mathbf{a} \parallel \mathbf{b} \) |
| Dot product with vectors scaled by \( s_1 \) and \( s_2 \) | \( s_1 \mathbf{a} \cdot s_2 \mathbf{b} = s_1 s_2 \mathbf{a} \cdot \mathbf{b} \) |
| Commutative law | \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \) |
| Distributive law | \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \) |
| Distributive law | \( (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d} \) |

Note: The distributive law for dot-products and cross-products is proved in [32, pgs. 23-24, 32-34].

2.9.2 Uses for the dot-product (·)

- Calculating an angle between two vectors [see equation (3) and example in Section 3.3]
- Determining when two vectors are perpendicular, e.g., \( \mathbf{a} \cdot \mathbf{b} = 0 \).
- Calculating a vector’s magnitude [see equation (4) and distance examples in Sections 3.2 and 3.3].
- Changing a vector equation into a scalar equation (see Homework 2.31).
- Calculating a unit vector in the direction of a vector \( \mathbf{v} \) [see equation (1)]
  \[
  \text{unitVector} = \frac{\mathbf{v}}{|\mathbf{v}|}
  \]
- Projection of a vector \( \mathbf{v} \) in the direction of \( \mathbf{b} \), defined as:
  \[
  \frac{\mathbf{v} \cdot \mathbf{b}}{|\mathbf{b}|}
  \]

2.9.3 Special case: Dot-products with orthogonal unit vectors

When \( \mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z \) are orthogonal unit vectors, it can be shown (see Homework 2.4)

\[
(a_x \mathbf{n}_x + a_y \mathbf{n}_y + a_z \mathbf{n}_z) \cdot (b_x \mathbf{n}_x + b_y \mathbf{n}_y + b_z \mathbf{n}_z) = a_x b_x + a_y b_y + a_z b_z
\]
2.9.4 Examples: Vector dot-products (\( \cdot \))

Shown below is how to use dot-products when vectors \( \vec{v} \) and \( \vec{w} \) are expressed in terms of orthogonal unit vectors \( \vec{n}_x, \vec{n}_y, \vec{n}_z \).

<table>
<thead>
<tr>
<th>( \vec{n}_x ) measure of ( \vec{v} )</th>
<th>( \vec{v} \cdot \vec{n}_x = 7 ) (measures how much of ( \vec{v} ) is in the ( \vec{n}_x ) direction).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{v} \cdot \vec{v} = 7^2 + 5^2 + 4^2 = 90 )</td>
<td>(</td>
</tr>
<tr>
<td>( \vec{w} \cdot \vec{w} = 2^2 + 3^2 + 2^2 = 17 )</td>
<td>(</td>
</tr>
</tbody>
</table>

Unit vector in the direction of \( \vec{v} \): \( \frac{\vec{v}}{|\vec{v}|} = \frac{7\vec{n}_x + 5\vec{n}_y + 4\vec{n}_z}{\sqrt{90}} \approx 0.738\vec{n}_x + 0.527\vec{n}_y + 0.422\vec{n}_z \)

Unit vector in the direction of \( \vec{w} \): \( \frac{\vec{w}}{|\vec{w}|} = \frac{2\vec{n}_x + 3\vec{n}_y + 2\vec{n}_z}{\sqrt{17}} \approx 0.485\vec{n}_x + 0.728\vec{n}_y + 0.485\vec{n}_z \)

\( \vec{v} \cdot \vec{w} = 7 \times 2 + 5 \times 3 + 4 \times 2 = 37 \) \( \angle(\vec{v}, \vec{w}) = \arccos\left(\frac{37}{\sqrt{90} \sqrt{17}}\right) \approx 0.33 \text{ rad} \approx 18.93^\circ \)

2.9.5 Dot-products to change vector equations to scalar equations (see Hw 1.31)

One way to form up to three linearly independent scalar equations from the vector equation \( \vec{v} = \vec{0} \) is by dot-multiplying \( \vec{v} = \vec{0} \) with three orthogonal unit vectors \( \vec{a}_1, \vec{a}_2, \vec{a}_3 \), i.e.,

\[
\text{if } \vec{v} = \vec{0} \Rightarrow \vec{v} \cdot \vec{a}_1 = 0 \quad \vec{v} \cdot \vec{a}_2 = 0 \quad \vec{v} \cdot \vec{a}_3 = 0
\]

Section 2.11.2 describes another way to form three different scalar equations from \( \vec{v} = \vec{0} \).

2.10 Vector cross product (\( \times \))

The cross product of a vector \( \vec{a} \) with a vector \( \vec{b} \) is defined in equation (5).

- \( |\vec{a}| \) and \( |\vec{b}| \) are the magnitudes of \( \vec{a} \) and \( \vec{b} \), respectively
- \( \theta \) is the smallest angle between \( \vec{a} \) and \( \vec{b} \) (\( 0 \leq \theta \leq \pi \))
- \( \hat{u} \) is the unit vector perpendicular to both \( \vec{a} \) and \( \vec{b} \)

The direction of \( \hat{u} \) is determined by the right-hand rule.

The right-hand rule is a convention like driving on the right-hand side of the road in North America. Until 1965, the Soviet Union used the left-hand rule.

Note: \( |\vec{a}| |\vec{b}| \sin(\theta) \) [the coefficient of \( \hat{u} \) in equation (5)] is inherently non-negative because \( \sin(\theta) \geq 0 \) since \( 0 \leq \theta \leq \pi \). Hence, \( |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin(\theta) \).

2.10.1 Properties of the cross-product (\( \times \))

Cross product with a zero vector \( \vec{a} \times \vec{0} = \vec{0} \)
Cross product of a vector with itself \( \vec{a} \times \vec{a} = \vec{0} \)
Cross product of parallel vectors \( \vec{a} \times \vec{b} = \vec{0} \) if \( \vec{a} \parallel \vec{b} \)
Cross product of scaled vectors \( s_1 \vec{a} \times s_2 \vec{b} = s_1 s_2 (\vec{a} \times \vec{b}) \)
Distributive law \( \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \) \( (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c} \)
Cross products are not associative \( \vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c} \)
Cross products are not commutative \( \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \)

Vector triple cross product \( \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b}) \)

A mnemonic for eqn (7) \( \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b}) \) is “back cab” - as in were you born in the back of a cab? Many proofs of this formula resolve \( \vec{a}, \vec{b}, \) and \( \vec{c} \) into orthogonal unit vectors (e.g., \( \vec{n}_x, \vec{n}_y, \vec{n}_z \)) and equate components.
2.10.2 Uses for the cross-product \((\times)\) in geometry, statics, motion analysis, . . .

- **Perpendicular** vectors, e.g., \(\vec{a} \times \vec{b}\) is perpendicular to both \(\vec{a}\) and \(\vec{b}\)
- **Moment** of a force or linear momentum, e.g., \(\vec{r} \times \vec{F}\) and \(\vec{r} \times m \vec{v}\)
- **Velocity/acceleration** formulas, e.g., \(\vec{v} = \vec{\omega} \times \vec{r}\) and \(\vec{a} = \vec{a} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})\)
- **Area of a triangle** whose sides have length \(|\vec{a}|\) and \(|\vec{b}|\)

The area of a parallelogram having sides of length \(|\vec{a}|\) and \(|\vec{b}|\) is \(|\vec{a} \times \vec{b}|\).

The area of a triangle \(\Delta\) is half the area of a parallelogram.

\[
\text{Vector area } \Delta(\vec{a}, \vec{b}) = \frac{1}{2} \vec{a} \times \vec{b} \quad \text{Scalar area } \Delta(\vec{a}, \vec{b}) = \frac{1}{2} |\vec{a} \times \vec{b}| \quad (8)
\]

Hw 2.14 and Section 3.3 show how cross-products calculate area (surveying).

- **Distance** \(d\) between a line \(L\) and a point \(Q\).

The line \(L\) (shown left) passes through point \(P\) and is parallel to the unit vector \(\hat{\vec{u}}\). The distance \(d\) between line \(L\) and a point \(Q\) can be calculated as

\[
d = |\vec{r}^{Q/P} \times \hat{\vec{u}}| = |\vec{r}^{Q/P}| \sin(\theta) \quad (9)
\]

Note: See example in Hw 1.26. Other distance calculations are in Sections 3.2 and 3.3.

2.10.3 Determinants and cross-products (with right-handed unit vectors)

When \(\hat{\vec{n}}_x, \hat{\vec{n}}_y, \hat{\vec{n}}_z\) are orthogonal unit vectors, it can be shown (Homework 2.13) the cross product of two vectors happens to be equal to the determinant of an associated matrix.

\[
\vec{a} = a_x \hat{\vec{n}}_x + a_y \hat{\vec{n}}_y + a_z \hat{\vec{n}}_z \quad \vec{b} = b_x \hat{\vec{n}}_x + b_y \hat{\vec{n}}_y + b_z \hat{\vec{n}}_z \quad \vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{\vec{n}}_x & \hat{\vec{n}}_y & \hat{\vec{n}}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = (a_y b_z - a_z b_y) \hat{\vec{n}}_x - (a_x b_z - a_z b_x) \hat{\vec{n}}_y + (a_x b_y - a_y b_x) \hat{\vec{n}}_z \quad (10)
\]

Examples: Vector cross-products \((\times)\) with determinants.

The following shows how to use cross-products with the vectors \(\vec{v}\) and \(\vec{w}\), each which is expressed in terms of the orthogonal unit vectors \(\hat{\vec{n}}_x, \hat{\vec{n}}_y, \hat{\vec{n}}_z\) shown to the right.

\[
\vec{v} = 7 \hat{\vec{n}}_x + 5 \hat{\vec{n}}_y + 4 \hat{\vec{n}}_z \quad \vec{w} = 2 \hat{\vec{n}}_x + 3 \hat{\vec{n}}_y + 2 \hat{\vec{n}}_z \quad \vec{v} \times \vec{w} = \det \begin{bmatrix} \hat{\vec{n}}_x & \hat{\vec{n}}_y & \hat{\vec{n}}_z \\ 7 & 5 & 4 \\ 2 & 3 & 2 \end{bmatrix} = -2 \hat{\vec{n}}_x - 6 \hat{\vec{n}}_y + 11 \hat{\vec{n}}_z
\]

Area from vectors \(\vec{v}\) and \(\vec{w}\): \(\Delta(\vec{v}, \vec{w}) = \frac{1}{2} |\vec{v} \times \vec{w}| = \frac{1}{2} \sqrt{(-2)^2 + (-6)^2 + 11^2} = \frac{\sqrt{161}}{2} \approx 6.344\).

Scalar triple product: \((2 \hat{\vec{n}}_x + 3 \hat{\vec{n}}_y + 4 \hat{\vec{n}}_z) \cdot (\vec{v} \times \vec{w}) = \det \begin{bmatrix} 2 & 3 & 4 \\ 7 & 5 & 4 \\ 2 & 3 & 2 \end{bmatrix} = 22\).
2.11 Optional: Scalar triple product \((\cdot \times \text{ or } \times \cdot)\)

The **scalar triple product** of vectors \(\vec{a}, \vec{b}, \vec{c}\) is the scalar defined in the various ways shown below.

\[
\text{ScalarTripleProduct} \triangleq \vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \cdot \vec{c} = \vec{b} \cdot \vec{c} \times \vec{a} = \vec{b} \times \vec{c} \cdot \vec{a} \quad (11)
\]

Although parentheses make equation (11) clearer, i.e., \(\text{ScalarTripleProduct} \triangleq \vec{a} \cdot (\vec{b} \times \vec{c})\), the parentheses are unnecessary because the cross product \(\vec{b} \times \vec{c}\) must be performed before the dot product for a sensible result to be produced.

2.11.1 Scalar triple product and the volume of a tetrahedron

For a tetrahedron whose sides are described by the vectors \(\vec{a}, \vec{b}, \vec{c}\) (sides of length \(|\vec{a}|, |\vec{b}|, |\vec{c}|\)), a geometrical interpretation of \(\vec{a} \cdot \vec{b} \times \vec{c}\) is the **volume of the parallelepiped**.

This formula helps calculate mass and volume (e.g., highway cut/fill calculations).

\[
\text{Tetrahedron Volume} = \frac{1}{6} \vec{a} \cdot \vec{b} \times \vec{c}
\]

2.11.2 \((\times \cdot)\) to change vector equations to scalar equations (see Hw 1.31)

Section 2.9.5 showed one method to form scalar equations from the vector equation \(\vec{v} = \vec{0}\).

A 2nd method expresses \(\vec{v}\) in terms of three non-coplanar (but not necessarily orthogonal or unit) vectors \(\vec{a}_1, \vec{a}_2, \vec{a}_3\), and writes the equally valid (but generally different) set of linearly independent scalar equations shown below.

Method 2: if \(\vec{v} = v_1 \vec{a}_1 + v_2 \vec{a}_2 + v_3 \vec{a}_3 = \vec{0}\) ⇒ \(v_1 = 0\), \(v_2 = 0\), \(v_3 = 0\)

Note: The proof that \(v_i = 0\) \((i = 1, 2, 3)\) follows directly by substituting \(\vec{v} = \vec{0}\) into equation (4.2).

2.12 Optional: Vectors vs. column matrices in the context of \(\vec{F} = m \vec{a}\)

In the context of matrices, the word “vector” describes a column matrix – which does **not** have direction. In the context of \(\vec{F} = m \vec{a}\), the word “vector” means something different. To associate direction to a column matrix, attach a basis as shown below.

\[
\vec{a}_x + 2\vec{a}_y + 3\vec{a}_z = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \vec{a}_x \\ \vec{a}_y \\ \vec{a}_z \end{bmatrix} = \begin{bmatrix} \vec{a}_x \\ \vec{a}_y \\ \vec{a}_z \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \hat{a}_x \\ \hat{a}_y \\ \hat{a}_z \end{bmatrix}_{\text{xyz}}
\]

Note: Although a vector can be represented with a column matrix and orthogonal unit vectors, it is not always desirable or efficient. Postponing resolution of vectors into orthogonal components allows maximum use of simplifying vector properties and avoids simplifications such as \(\sin^2(\theta) + \cos^2(\theta) = 1\) (see Homework 2.9). Words such as “vector” require context. Some English words are contranyms (opposite meanings) such as “fast” and “bolt” (move quickly or fasten).